UNIQUE EXPANSIONS OF REAL NUMBERS

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Dedicated to the memory of Paul Erdős

Abstract. It was discovered some years ago that there exist non-integer real numbers $q > 1$ for which only one sequence (c_i) of integers $c_i \in [0, q)$ satisfies the equality $\sum_{i=1}^{\infty} c_i q^{-i} = 1$. The set of such "univoque numbers" has a rich topological structure, and its study revealed a number of unexpected connections with measure theory, fractals, ergodic theory and Diophantine approximation.

In this paper we consider for each fixed $q > 1$ the set $\mathcal{U}_q$ of real numbers x having a unique representation of the form $\sum_{i=1}^{\infty} c_i q^{-i} = x$ with integers c_i belonging to $[0, q)$. We carry out a detailed topological study of these sets. For instance, we characterize their closures, and we determine those bases q for which $\mathcal{U}_q$ is closed or even a Cantor set. We also study the set $\mathcal{U}'_q$ consisting of all sequences (c_i) of integers $c_i \in [0, q)$ such that $\sum_{i=1}^{\infty} c_i q^{-i} \in \mathcal{U}_q$. We determine the numbers $r > 1$ for which the map $q \mapsto \mathcal{U}'_q$ (defined on $(1, \infty)$) is constant in a neighborhood of r and the numbers $q > 1$ for which $\mathcal{U}'_q$ is a subshift or a subshift of finite type.

1. Introduction and statement of the main results

Following a seminal paper of Rényi [R] many works were devoted to probabilistic, measure theoretical and number theoretical aspects of representations for real numbers in non-integer bases; see, e.g., Frougny and Solomyak [FS], Pethő and Tichy [PT], Schmidt [Sc]. A new research field was opened when Erdős, Horváth and Joó [EHJ] discovered continuum many real numbers $q > 1$ for which only one sequence $(c_i) = c_1c_2\dots$ of integers c_i belonging to $[0, q)$ satisfies the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = 1.$$

(They considered the case $1 < q < 2$.) Subsequently, the set $\mathcal{U}$ of all such *univoque numbers* $q > 1$ was characterized lexicographically in [EJK1, KL3], its smallest element was determined in [KL1], and its topological structure was described in [KL3]. On the other hand, the investigation of numbers $q > 1$ for which there exist continuum many such sequences, including sequences containing all possible finite variations of the integers $c \in [0, q)$, revealed close connections to Diophantine approximations; see, e.g., [EJK1, EJK3, EK, KLP], Borwein and Hare [BH1, BH2], Komatsu [K], and Sidorov [Si1].

For any fixed real number $q > 1$, we may also introduce the set $\mathcal{U}_q$ of real numbers x for which exactly one sequence (c_i) of integers $c_i \in [0, q)$ satisfies the equality

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

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If q is an integer, these sets are well-known. However, their structure is more complex if q is a non-integer, see, e.g., Daróczy and Kátai [DK1, DK2], Glendinning and Sidorov [GS], and Kallós [K1, K2]. The purpose of this paper is to give a complete topological description of the sets $\mathcal{U}_q$: they have a different nature for different classes of the numbers q. Our investigations also provide new results concerning the set $\mathcal{U}$ of univoque numbers. For instance, we determine for each $n \in \mathbb{N} := \{1, 2, \dots\}$ the smallest element of $\mathcal{U} \cap (n, n+1)$ and we continue the study of the topological structure of $\mathcal{U}$, started in [KL3]. In order to state our results we need to introduce some notation and terminology.

In this paper a *sequence* always means an element of the set $\{0, 1, 2, \dots\}^{\mathbb{N}}$. A sequence is called *infinite* if it contains infinitely many nonzero elements; otherwise it is called *finite*. We use systematically the lexicographical order between sequences: we write $(a_i) < (b_i)$ or $(b_i) > (a_i)$ if there exists an index $n \in \mathbb{N}$ such that $a_i = b_i$ for $i < n$ and $a_n < b_n$. We also equip for each $n \in \mathbb{N}$ the set $\{0, 1, 2, \dots\}^n$ of *blocks of length* n with the lexicographical order.

Given a real number $q > 1$, an *expansion in base* q (or simply *expansion*) of a real number x is a sequence (c_i) such that

$$0 \le c_i < q \text{ for all } i \ge 1, \quad \text{and} \quad x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}.$$

If a real number x has an expansion in base q, then x must belong to the interval

$$J_q := \left[0, \frac{\lceil q \rceil - 1}{q - 1}\right]$$

where $\lceil q \rceil$ denotes the smallest integer larger than or equal to q. Note that $1 \in J_q$.

A *sequence* (c_i) such that $0 \le c_i < q$ for all $i \ge 1$ is called *univoque* in base q if

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

is an element of $\mathcal{U}_q$.

The *greedy* expansion $(b_i(x, q)) = (b_i(x)) = (b_i)$ of a number $x \in J_q$ in base q is the largest expansion of x in lexicographical order. It is well-known that the greedy expansion of any $x \in J_q$ exists; [R, P, EJK2]. A *sequence* (b_i) is called *greedy* in base q if (b_i) is the greedy expansion of

$$x = \sum_{i=1}^{\infty} \frac{b_i}{q^i}.$$

The *quasi-greedy* expansion $(a_i(x, q)) = (a_i(x)) = (a_i)$ of a number $x \in J_q \setminus \{0\}$ in base q is the largest *infinite* expansion of x in lexicographical order. Observe that we have to exclude the number 0 since there do not exist infinite expansions of $x = 0$ at all. On the other hand, the largest infinite expansion of any $x \in J_q \setminus \{0\}$ exists, as we shall prove in the next section. In order to simplify some statements below, the quasi-greedy expansion of the number $0 \in J_q$ is defined to be $0^\infty = 00\dots$. Note that this is the only expansion of $x = 0$. A *sequence* (a_i) is called *quasi-greedy* in base q if (a_i) is the quasi-greedy expansion of

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

We usually denote the quasi-greedy expansion $(a_i(1, q))$ of the number 1 in base q by $(\alpha_i(q)) = (\alpha_i)$. To stress that the quasi-greedy expansion of 1 in base q is

given by (α_i), we sometimes write $q \sim (\alpha_i)$. This notation is particularly convenient in Section 6 where we consider expansions $(\alpha_i(q))$ for different values of q simultaneously.

Since $\alpha_1 = \lceil q \rceil - 1$ (as we shall see in the next section), the digits c_i of an expansion (c_i) belong to $A := \{0, \dots, \alpha_1\}$ for all $i \geq 1$. Hence we consider expansions with coefficients or digits in the alphabet A of numbers $x \in [0, \alpha_1/(q-1)]$.

The greedy expansion of a number $x \in J_q \setminus \{0\}$ coincides with the quasi-greedy expansion if and only if the greedy expansion of x is infinite. If the greedy expansion (b_i) of $x \in J_q \setminus \{0\}$ is finite and b_n is its last nonzero element, then the quasi-greedy expansion of x is given by

$$(a_i) = b_1 \dots b_{n-1} b_n^- \alpha_1 \alpha_2 \dots, \quad \text{where} \quad b_n^- := b_n - 1.$$

For instance, if q equals the *golden ratio* $G := (1+\sqrt{5})/2$ and $x = q^{-1} + q^{-2} + q^{-3}$, then $(\alpha_i) = (10)^\infty$, $(b_i(x)) = 1110^\infty$, and $(a_i(x)) = 1(10)^\infty$.

Of course, whether a sequence is univoque, greedy or quasi-greedy depends on the base q. However, when q is understood from the context, we simply speak of univoque sequences and (quasi)-greedy sequences. Furthermore, we shall write $\overline{c} := \alpha_1 - c \, (c \in A)$, unless stated otherwise. We shall also write $\overline{c_1 \dots c_n}$ instead of $\overline{c_1} \dots \overline{c_n}$ and $\overline{c_1 c_2 \dots}$ instead of $\overline{c_1}\,\overline{c_2} \dots$ $(c_i \in A, i \geq 1)$. Sometimes we refer to $\overline{c_1 c_2 \dots}$ as the *conjugate* of an expansion (c_i). Finally, we set $c^+ := c + 1$ $(c \in A \setminus \{\alpha_1\})$ and $c^- := c - 1$ $(c \in A \setminus \{0\})$.

The following important theorem, which is essentially due to Parry [P](see also [DK1, DK2]), plays a crucial role in the proof of our main results:

Theorem 1.1. *Fix $q > 1$.*

(i) *A sequence $(b_i) = b_1 b_2 \dots \in \{0, \dots, \alpha_1\}^{\mathbb{N}}$ is greedy if and only if*

$$b_{n+1} b_{n+2} \dots < \alpha_1 \alpha_2 \dots \quad \textit{whenever} \quad b_n < \alpha_1.$$

(ii) *A sequence $(c_i) = c_1 c_2 \dots \in \{0, \dots, \alpha_1\}^{\mathbb{N}}$ is univoque if and only if*

$$c_{n+1} c_{n+2} \dots < \alpha_1 \alpha_2 \dots \quad \textit{whenever} \quad c_n < \alpha_1$$

and

$$\overline{c_{n+1} c_{n+2} \dots} < \alpha_1 \alpha_2 \dots \quad \textit{whenever} \quad c_n > 0.$$

Note that (α_i) is the unique expansion of 1 in base q if and only if $q \in \mathcal{U}$. Hence, replacing the sequence (c_i) in Theorem 1.1 (ii) with the sequence (α_i), one obtains a characterization of $\mathcal{U}$.

Recently, the authors of [KL3] studied the topological structure of the set $\mathcal{U}$. In particular, they showed that $\mathcal{U}$ is *not* closed and they characterized its closure $\overline{\mathcal{U}}$:

Theorem 1.2. *A real number $q > 1$ belongs to $\overline{\mathcal{U}}$ if and only if the quasi-greedy expansion (α_i) of the number 1 in base q satisfies*

$$\overline{\alpha_{k+1} \alpha_{k+2} \dots} < \alpha_1 \alpha_2 \dots \quad \textit{for all} \quad k \geq 1.$$

It is possible to give a similar description of the set $\overline{\mathcal{U}}$ in words: $q > 1$ belongs to $\overline{\mathcal{U}}$ if and only if $(\alpha_i(q))$ is the unique *infinite* expansion of the number 1 in base q (see Corollary 5.4).

Remarks.

- Recall that the number q is not allowed in any expansion in base q if q is an integer. Our choice of the digit set simplifies some statements. For example it will follow from the theorems below that

$$\mathcal{U}_q = \overline{\mathcal{U}_q} \quad \Longleftrightarrow \quad q \in (1, \infty) \setminus \overline{\mathcal{U}}$$

where $\overline{\mathcal{U}_q}$ denotes the closure of $\mathcal{U}_q$.

- In the definition of $\mathcal{U}$ given in [KL3] the integers $2, 3, \ldots$ were excluded. However, its closure is the same as the set $\overline{\mathcal{U}}$ defined in this paper. As a consequence, Theorem 1.2 still holds in our set-up.
- It follows from Theorems 1.1, 1.2 and Proposition 2.3 below that the quasi-greedy expansion of 1 in base q is periodic for each $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. For instance, if $n \geq 2$, then $(1^n0)^\infty = (\alpha_i(q))$ for some $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$.

For any fixed $q > 1$, we introduce the set $\mathcal{V}_q$, defined by

$$\mathcal{V}_q = \left\{ x \in J_q : \overline{a_{n+1}(x)a_{n+2}(x)\ldots} \leq \alpha_1\alpha_2\ldots \quad \text{whenever} \quad a_n(x) > 0 \right\}.$$

It follows from Theorem 1.1 that $\mathcal{U}_q \subset \mathcal{V}_q$ for all $q > 1$.

Now we are ready to state our main results.

Theorem 1.3. *Suppose that $q \in \overline{\mathcal{U}}$. Then*

(i) $\overline{\mathcal{U}_q} = \mathcal{V}_q$;
(ii) $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ *and* $\mathcal{V}_q \setminus \mathcal{U}_q$ *is dense in* $\mathcal{V}_q$;
(iii) *if $q \in \mathcal{U}$, then each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly 2 expansions;*
(iv) *if $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions.*

Remarks.

- The proof of part (ii) yields the following more precise results where for $q \in \overline{\mathcal{U}}$ we set

$$A_q = \{x \in \mathcal{V}_q \setminus \mathcal{U}_q : x \text{ has a finite greedy expansion}\}$$

  and

$$B_q = \{x \in \mathcal{V}_q \setminus \mathcal{U}_q : x \text{ has an infinite greedy expansion}\} :$$

  - If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then both A_q and B_q are countably infinite and dense in $\mathcal{V}_q$. Moreover, the greedy expansion of a number $x \in B_q$ ends with $\overline{\alpha_1\alpha_2\ldots}$, and

$$B_q = \{\alpha_1/(q-1) - x : x \in A_q\}.$$

  - If $q \in \{2, 3, \ldots\}$, then $B_q = \varnothing$.
- For each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$, the proof of parts (iii) and (iv) also provides the list of all expansions of x in terms of its greedy expansion.

Our next goal is to describe the relationship between the sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ in case $q \notin \overline{\mathcal{U}}$. To this end, we introduce the set $\mathcal{V}$, consisting of those numbers $q > 1$, for which the quasi-greedy expansion (α_i) of the number 1 in base q satisfies

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} \leq \alpha_1\alpha_2\ldots \quad \text{for all} \quad k \geq 1.$$

It follows from Theorem 1.2 that $\mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$. The following results combined with Theorem 1.3 imply that $\mathcal{U}_q$ is closed if $q \notin \overline{\mathcal{U}}$ and that the set $\mathcal{V}_q$ is closed for all $q > 1$.

Theorem 1.4. *Suppose that $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Then*

(i) *the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are closed;*
(ii) $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ *and $\mathcal{V}_q \setminus \mathcal{U}_q$ is a discrete set, dense in $\mathcal{V}_q$;*
(iii) *each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions and a finite greedy expansion.*

Remark. Our proof also provides the list of all expansions of all elements $x \in \mathcal{V}_q \setminus \mathcal{U}_q$.

Theorem 1.5. *Suppose that $q \in (1, \infty) \setminus \mathcal{V}$. Then*

$$\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q.$$

Remarks.

- In view of the above results, Theorem 1.1 already gives us a lexicographical characterization of $\overline{\mathcal{U}_q}$ if $q \in (1,\infty)\setminus\overline{\mathcal{U}}$ because in this case $\mathcal{U}_q$ is closed.
- It is well-known that the set $\mathcal{U}$ has Lebesgue measure zero; [EHJ, KK]. In [KL3] it was shown that the set $\overline{\mathcal{U}}\setminus\mathcal{U}$ is countably infinite. It follows from the above results that $\mathcal{U}_q$ is closed for almost every $q > 1$.
- Let $q > 1$ be a non-integer. In [DDV] it has been proved that almost every $x \in J_q$ has a continuum of expansions in base q (see also [Si2]). It follows from the above results that the set $\overline{\mathcal{U}_q}$ has Lebesgue measure zero. Hence the set $\mathcal{U}_q$ is nowhere dense.
- Let $q > 1$ be an integer. In this case the quasi-greedy expansion of 1 in base q is given by $(\alpha_i) = \alpha_1^\infty = (q-1)^\infty$. Moreover, the set $J_q \setminus \mathcal{U}_q$ is countably infinite and each element in $J_q \setminus \mathcal{U}_q$ has only two expansions, one of them being finite while the other one ends with an infinite string of $(q-1)$'s.
- In [KL1] it was shown that the smallest element of $\mathcal{U}$ is given by $q' \approx 1.787$, and the unique expansion of 1 in base q' is the truncated Thue–Morse sequence $(\tau_i) = \tau_1\tau_2\ldots = 11010011\ldots$, which can be defined recursively by setting $\tau_{2^N} = 1$ for $N = 0,1,2,\ldots$ and
$$\tau_{2^N+i} = 1-\tau_i \quad \text{for } 1 \le i < 2^N,\ N = 1,2,\ldots.$$
Subsequently, Glendinning and Sidorov [GS] proved that $\mathcal{U}_q$ is countable [1] if $1 < q < q'$ and has the cardinality of the continuum if $q \in [q',2)$. Moreover, they showed that $\mathcal{U}_q$ is a set of positive Hausdorff dimension if $q' < q < 2$, and they described a method to compute its Hausdorff dimension (see also [DK2, K1, K2]).

In the following theorem we characterize those $q > 1$ for which $\mathcal{U}_q$ or $\overline{\mathcal{U}_q}$ is a Cantor set, i.e., a nonempty closed set having neither interior nor isolated points. We recall from [KL3] that

- $\mathcal{V}$ is closed and $\mathcal{U}$ is closed from above [2],
- $|\overline{\mathcal{U}}\setminus\mathcal{U}| = \aleph_0$ and $\overline{\mathcal{U}}\setminus\mathcal{U}$ is dense in $\overline{\mathcal{U}}$,
- $|\mathcal{V}\setminus\overline{\mathcal{U}}| = \aleph_0$ and $\mathcal{V}\setminus\overline{\mathcal{U}}$ is a discrete set, dense in $\mathcal{V}$.

Since the set $(1,\infty)\setminus\mathcal{V}$ is open, we can write $(1,\infty)\setminus\mathcal{V}$ as the union of countably many disjoint open intervals (q_1,q_2): its connected components. Let us denote by L and R the set of left (respectively right) endpoints of the intervals (q_1,q_2).

Theorem 1.6.

(i) $L = \mathbb{N}\cup(\mathcal{V}\setminus\mathcal{U})$ *and* $R = \mathcal{V}\setminus\overline{\mathcal{U}}$. *Hence* $R \subset L$, *and*
$$(1,\infty)\setminus\overline{\mathcal{U}} = \cup(q_1,q_2]$$
where the union runs over the connected components (q_1,q_2) *of* $(1,\infty)\setminus\mathcal{V}$.

(ii) *If* $q \in \{2,3,\ldots\}$, *then neither* $\mathcal{U}_q$ *nor* $\overline{\mathcal{U}_q}$ *is a Cantor set.*

(iii) *If* $q \in \overline{\mathcal{U}}\setminus\mathbb{N}$, *then* $\mathcal{U}_q$ *is not a Cantor set, but its closure* $\overline{\mathcal{U}_q}$ *is a Cantor set.*

(iv) *If* $q \in (q_1,q_2]$, *where* (q_1,q_2) *is a connected component of* $(1,\infty)\setminus\mathcal{V}$, *then the closed set* $\mathcal{U}_q$ *is a Cantor set if and only if* $q_1 \in \{3,4,\ldots\}\cup(\overline{\mathcal{U}}\setminus\mathcal{U})$. *Moreover, if* $q_1 \in \{1,2\}\cup(\mathcal{V}\setminus\overline{\mathcal{U}})$, *then the isolated points of* $\mathcal{U}_q$ *form a dense subset of* $\mathcal{U}_q$.

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[1] Here and in the sequel, we call a set *countable* if it is either finite or countably infinite.

[2] We call a set $X \subset \mathbb{R}$ *closed from above* (*closed from below*) if for each $x \in \mathbb{R}\setminus X$ there exists a number $\delta > 0$ such that $[x,x+\delta)\cap X = \varnothing$ ($(x-\delta,x]\cap X = \varnothing$).

Remark. We also describe the set of endpoints of the connected components (p_1, p_2) of $(1,\infty)\setminus\overline{\mathcal{U}}$: denoting by L' and R' the set of left (respectively right) endpoints of the intervals (p_1, p_2), we have

$$L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad R' \subset \mathcal{U}.$$

This enables us to determine the condensation points of $\mathcal{U}_q$ for each $q > 1$; see the remarks at the end of Section 6.

The ideas leading to the above theorem result in a new characterization of *stable bases*, introduced and investigated by Daróczy and Kátai ([DK1, DK2]). Let us denote by $\mathcal{U}_q'$ and $\mathcal{V}_q'$ the sets of quasi-greedy expansions in base q of all numbers $x \in \mathcal{U}_q$ and $x \in \mathcal{V}_q$ respectively. Note that $\mathcal{U}_q'$ is simply the set of univoque sequences in base q.

If $1 < q < r$ and $\mathcal{U}_q' = \mathcal{U}_r'$, then $\lceil q \rceil = \lceil r \rceil$ and $\mathcal{U}_q' = \mathcal{U}_t'$ for each $t \in (q, r)$, as follows from Theorem 1.1 and Proposition 2.3 below. For this reason, we call a number $q > 1$ *stable from above* (respectively *stable from below*) if there exists a number $s > q$ (respectively $1 < s < q$) such that

$$\mathcal{U}_q' = \mathcal{U}_s'.$$

We call a number $q > 1$ *stable* if it is stable from above and from below. Finally, we say that an interval $I \subset (1,\infty)$ is a *stability interval* if $\mathcal{U}_q' = \mathcal{U}_s'$ for all $q, s \in I$.

Theorem 1.7. *The maximal stability intervals are given by the singletons $\{q\}$ where $q \in \overline{\mathcal{U}}$ and the intervals $(q_1, q_2]$ where (q_1, q_2) is a connected component of $(1,\infty)\setminus\mathcal{V}$. Moreover, if $q_1 \in \mathcal{V}\setminus\mathcal{U}$, then*

$$\mathcal{U}_q' = \mathcal{V}_{q_1}' \quad \textit{for all} \quad q \in (q_1, q_2].$$

Remark. The proof of Theorem 1.7 yields a new characterization of the sets $\overline{\mathcal{U}}$ and $\mathcal{V}$ (see Proposition 6.9).

We recall (see, e.g., [LM]) that a set $S \subset \{0,\dots,\alpha_1\}^{\mathbb{N}}$ is called a *subshift* if there exists a set $\mathcal{F}(S) \subset \cup_{k=1}^{\infty}\{0,\dots,\alpha_1\}^k$ such that a sequence $(c_i) \in \{0,\dots,\alpha_1\}^{\mathbb{N}}$ belongs to S if and only if none of the blocks $c_{i+1}\dots c_{i+n}$ $(i \ge 0, n \ge 1)$ belongs to $\mathcal{F}(S)$. A subshift S is called a *subshift of finite type* if one can choose $\mathcal{F}(S)$ to be finite. We endow the set $\{0,\dots,\alpha_1\}^{\mathbb{N}}$ with the topology of coordinate-wise convergence.

Theorem 1.8. *Let $q > 1$ be a real number. The following statements are equivalent.*

(i) *$q \in (1,\infty)\setminus\overline{\mathcal{U}}$.*
(ii) *$\mathcal{U}_q'$ is a subshift of finite type.*
(iii) *$\mathcal{U}_q'$ is a subshift.*
(iv) *$\mathcal{U}_q'$ is a closed subset of $\{0,\dots,\alpha_1\}^{\mathbb{N}}$.*

Finally, we determine the cardinality of $\mathcal{U}_q$ for all $q > 1$. We recall that for $q \in (1,2)$ this has already been done by Glendinning and Sidorov ([GS]), using a different method. Let q'' be the smallest element of $\mathcal{U}\cap(2,3)$. It was shown in [KL2] that the unique expansion of 1 in base q'' is given by $(\lambda_i) = \lambda_1\lambda_2\dots = 21020121\dots$, where $\lambda_i = \tau_i + \tau_{2i-1}$, $i \ge 1$.

Theorem 1.9. *Let $q > 1$ be a real number.*

(i) *If $q \in (1, G]$, then $\mathcal{U}_q$ consists merely of the endpoints of J_q.*
(ii) *If $q \in (G, q') \cup (2, q'')$, then $|\mathcal{U}_q| = \aleph_0$.*
(iii) *If $q \in [q', 2] \cup [q'', \infty)$, then $|\mathcal{U}_q| = 2^{\aleph_0}$.*

Remarks.

- We also determine the unique expansion of 1 in base $q^{(n)}$ for $n \in \{3, 4, \ldots\}$ where $q^{(n)}$ denotes the smallest element of $\mathcal{U} \cap (n, n+1)$; see the remarks at the end of Section 6.
- A generalization of Theorem 1.9 can be found in [DV2].

For the reader's convenience we recall some properties of quasi-greedy expansions in the next section. These properties are also stated in [BK] and are closely related to some important results, first established in the seminal works by Rényi [R] and Parry [P]. In Section 3 we derive some preliminary lemmas needed for the proof of our main results. Section 4 is then devoted to the proof of Theorem 1.3. Theorems 1.4 and 1.5 are proved in Section 5, and our final Theorems 1.6, 1.7, 1.8 and 1.9 are established in Section 6.

2. Quasi-greedy expansions

Let $q > 1$ be a real number and let $m = \lceil q \rceil - 1$. In the previous section we defined the quasi-greedy expansion of $x \in (0, m/(q-1)]$ as its largest infinite expansion in base q. In order to prove that this notion is well-defined, we introduce the *quasi-greedy algorithm*: if for some $n \in \mathbb{N}$, $a_i(x) = a_i$ is already defined for i with $1 \le i < n$ (no condition if $n = 1$), then $a_n(x) = a_n$ is the largest element of the digit set $A = \{0, \ldots, m\}$ such that

$$\sum_{i=1}^{n} \frac{a_i}{q^i} < x.$$

Of course, this definition is only meaningful if $x > 0$. In the following proposition we show that this algorithm generates an expansion of x for all $x \in J_q \setminus \{0\}$. It follows that the quasi-greedy expansion of $x \in J_q \setminus \{0\}$ is obtained by performing the quasi-greedy algorithm.

Proposition 2.1. *Let $x \in (0, m/(q-1)]$. Then*

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

Proof. If $x = m/(q-1)$, then the quasi-greedy algorithm provides $a_i = m$ for all $i \ge 1$ and the desired equality follows.

Suppose that $x \in (0, m/(q-1))$. Then, by definition of the quasi-greedy algorithm, there exists an index n such that $a_n < m$.

First assume that $a_n < m$ for infinitely many n. For any such n, we have by definition

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{q^i} \le \frac{1}{q^n}.$$

Letting $n \to \infty$, we obtain

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i}.$$

Next assume there exists a largest n such that $a_n < m$. Then

$$\sum_{i=1}^{n} \frac{a_i}{q^i} + \sum_{i=n+1}^{N} \frac{m}{q^i} < x \le \sum_{i=1}^{n} \frac{a_i}{q^i} + \frac{1}{q^n},$$

for each $N > n$. Hence

$$\sum_{i=n+1}^{\infty} \frac{m}{q^i} \le x - \sum_{i=1}^{n} \frac{a_i}{q^i} \le \frac{1}{q^n}.$$

Note that

$$\frac{1}{q^n} \le \sum_{i=n+1}^{\infty} \frac{m}{q^i}$$

for any $q > 1$, and

$$\frac{1}{q^n} = \sum_{i=n+1}^{\infty} \frac{m}{q^i}$$

if and only if $q = m + 1$. Hence q is an integer, and

$$x = \sum_{i=1}^{n} \frac{a_i}{q^i} + \sum_{i=n+1}^{\infty} \frac{m}{q^i} = \sum_{i=1}^{\infty} \frac{a_i}{q^i}. \qquad \square$$

Now we consider the quasi-greedy expansion $(\alpha_i(q)) = (\alpha_i)$ of $x = 1$. Note that $\alpha_1 = m = \lceil q \rceil - 1$ by definition of the quasi-greedy algorithm.

Lemma 2.2. *For each $n \ge 1$, the inequality*

$$\sum_{i=1}^{n} \frac{\alpha_i}{q^i} + \frac{1}{q^n} \ge 1 \tag{2.1}$$

holds.

Proof. The proof is by induction on n. For $n = 1$, the inequality holds because $\alpha_1 + 1 \ge q$. Assume the inequality is valid for some $n \in \mathbb{N}$. If $\alpha_{n+1} < \alpha_1$, then (2.1) with $n + 1$ instead of n follows from the definition of the quasi-greedy algorithm. If $\alpha_{n+1} = \alpha_1$, then the same conclusion follows from the induction hypothesis and the inequality $\alpha_1 + 1 \ge q$. $\square$

Proposition 2.3. *The map $q \mapsto (\alpha_i(q))$ is a strictly increasing bijection from the open interval $(1, \infty)$ onto the set of all* infinite *sequences (α_i), satisfying*

$$\alpha_{k+1}\alpha_{k+2}\ldots \le \alpha_1\alpha_2\ldots \quad \textit{for all} \quad k \ge 1. \tag{2.2}$$

Proof. By definition of the quasi-greedy algorithm, the map $q \mapsto (\alpha_i(q))$ is strictly increasing. Fix $q > 1$ and $k \in \mathbb{N}$. By the preceding lemma we have

$$\sum_{i=1}^{k} \frac{\alpha_i(q)}{q^i} + \frac{1}{q^k} \ge \sum_{i=1}^{\infty} \frac{\alpha_i(q)}{q^i}$$

whence

$$\sum_{i=1}^{\infty} \frac{\alpha_{k+i}(q)}{q^i} \le 1. \tag{2.3}$$

If we had $(\alpha_{k+i}(q)) > (\alpha_i(q))$, then by Lemma 2.2,

$$\sum_{i=1}^{n} \frac{\alpha_{k+i}(q)}{q^i} \ge 1$$

for some $n \in \mathbb{N}$, which contradicts (2.3) because $(\alpha_{k+i}(q))$ is infinite.

Conversely, let (α_i) be an infinite sequence satisfying (2.2). Solving the equation

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} = 1,$$

we obtain a unique number $q > 1$. Note that $0 \le \alpha_n \le \alpha_1 < q$ for $n \ge 1$. In order to prove that $(\alpha_i) = (\alpha_i(q))$, it suffices to show that for each $n \ge 1$, the inequality

$$\sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} \le \frac{1}{q^n}$$

holds. Starting with $k_0 := n$ and using (2.2), we *try* to define a sequence

$$k_0 < k_1 < \cdots$$

satisfying for $j = 1, 2, \dots$ the conditions

$$\alpha_{k_{j-1}+i} = \alpha_i \quad \text{for } i = 1, \dots, k_j - k_{j-1} - 1 \quad \text{and } \alpha_{k_j} < \alpha_{k_j - k_{j-1}}.$$

If we obtain in this way an infinite number of indices, then we have

$$\begin{aligned}\sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} &\le \sum_{j=1}^{\infty} \left(\left(\sum_{i=1}^{k_j-k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} \right) - \frac{1}{q^{k_j}} \right) \\ &< \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) = \frac{1}{q^n}.\end{aligned}$$

If we only obtain a finite number of indices, then there exists a least nonnegative integer N ($N = 0$ is possible) such that $(\alpha_{k_N+i}) = (\alpha_i)$ and we have

$$\begin{aligned}\sum_{i=n+1}^{\infty} \frac{\alpha_i}{q^i} &\le \sum_{j=1}^{N} \left(\left(\sum_{i=1}^{k_j-k_{j-1}} \frac{\alpha_i}{q^{k_{j-1}+i}} \right) - \frac{1}{q^{k_j}} \right) + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{k_N+i}} \\ &\le \sum_{j=1}^{N} \left(\frac{1}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) + \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{k_N+i}} = \frac{1}{q^n}.\end{aligned}$$

□

The proof of the following propositions is almost identical to the proof of Proposition 2.3 and is therefore omitted.

Proposition 2.4. *For each $q > 1$, the map $x \mapsto (a_i(x))$ is a strictly increasing bijection from $(0, \alpha_1/(q-1)]$ onto the set of all* infinite *sequences (a_i), satisfying*

$$0 \le a_n \le \alpha_1 \quad \textit{for all} \quad n \ge 1$$

and

$$a_{n+1}a_{n+2}\dots \le \alpha_1\alpha_2\dots \quad \textit{whenever} \quad a_n < \alpha_1. \tag{2.4}$$

Proposition 2.5. *For each $q > 1$, the map $x \mapsto (b_i(x))$ is a strictly increasing bijection from $[0, \alpha_1/(q-1)]$ onto the set of all sequences (b_i), satisfying*

$$0 \le b_n \le \alpha_1 \quad \textit{for all} \quad n \ge 1$$

and

$$b_{n+1}b_{n+2}\dots < \alpha_1\alpha_2\dots \quad \textit{whenever} \quad b_n < \alpha_1. \tag{2.5}$$

Remarks.

- A sequence (c_i) is univoque if and only if (c_i) is greedy and $(\alpha_1 - c_i)$ is greedy. Hence Theorem 1.1 is a consequence of Proposition 2.5.
- The greedy expansion (b_i) of $x \in J_q$ is generated by the *greedy algorithm*: if for some $n \in \mathbb{N}$, b_i is already defined for i with $1 \le i < n$ (no condition if $n = 1$), then b_n is the largest element of A such that

$$\sum_{i=1}^{n} \frac{b_i}{q^i} \le x.$$

The proof of this assertion goes along the same lines as the proof of Proposition 2.1.

3. Some preliminary results

Throughout this section, $q > 1$ is an arbitrary but fixed real number.

Lemma 3.1. *Let $(d_i) = d_1 d_2 \dots$ be a greedy or quasi-greedy sequence. Then for all $N \geq 1$ the truncated sequence $d_1 \dots d_N 0^\infty$ is greedy.*

Proof. If $(d_i) = 0^\infty$, then there is nothing to prove. If $(d_i) \neq 0^\infty$, then the statement follows from Propositions 2.4 and 2.5. □

Lemma 3.2. *Let $(b_i) \neq \alpha_1^\infty$ be a greedy sequence and let N be a positive integer. Then there exists a greedy sequence $(c_i) > (b_i)$ such that*

$$c_1 \dots c_N = b_1 \dots b_N.$$

Proof. Since $(b_i) \neq \alpha_1^\infty$, it follows from (2.5) that $b_n < \alpha_1$ for infinitely many n. Hence we may assume, by enlarging N if necessary, that $b_N < \alpha_1$. Let

$$I = \{i \in \mathbb{N} : 1 \leq i \leq N \text{ and } b_i < \alpha_1\} =: \{i_1, \dots, i_n\}.$$

Note that for $i_r \in I$,

$$\sum_{j=1}^{\infty} \frac{b_{i_r+j}}{q^j} = \sum_{j=1}^{N-i_r} \frac{b_{i_r+j}}{q^j} + \frac{1}{q^{N-i_r}} \sum_{i=1}^{\infty} \frac{b_{N+i}}{q^i} < 1$$

because (b_i) is greedy and $b_{i_r} < \alpha_1$. For each $r \in \{1, \dots, n\}$, choose y_r such that

$$\sum_{i=1}^{\infty} \frac{b_{N+i}}{q^i} < y_r \leq \frac{\alpha_1}{q-1} \tag{3.1}$$

and

$$\sum_{j=1}^{N-i_r} \frac{b_{i_r+j}}{q^j} + \frac{y_r}{q^{N-i_r}} < 1. \tag{3.2}$$

Let $y = \min\{y_1, \dots, y_n\}$ and denote the greedy expansion of y by $d_1 d_2 \dots$. Finally, let $(c_i) = b_1 \dots b_N d_1 d_2 \dots$. From (3.1) we infer that $(c_i) > (b_i)$. It remains to show that (c_i) is a greedy sequence, i.e., we need to show that

$$\sum_{i=1}^{\infty} \frac{c_{j+i}}{q^i} < 1 \quad \text{whenever} \quad c_j < \alpha_1. \tag{3.3}$$

If $c_j < \alpha_1$ and $j \leq N$, then (3.3) follows from (3.2). If $c_j < \alpha_1$ and $j > N$, then (3.3) follows from the fact that (d_i) is a greedy sequence. □

Lemma 3.3. *If $(b_i) \neq \alpha_1^\infty$ is a greedy sequence, then there exists a sequence $1 \leq n_1 < n_2 < \cdots$ such that for each $i \geq 1$,*

$$b_{n_i} < \alpha_1 \quad \textit{and} \quad b_{m+1} \dots b_{n_i} < \alpha_1 \dots \alpha_{n_i-m} \quad \textit{if } m < n_i \textit{ and } b_m < \alpha_1. \tag{3.4}$$

Proof. We define a sequence $(n_i)_{i \geq 1}$ satisfying the requirements by induction.

Let r be the least positive integer for which $b_r < \alpha_1$. Then, (3.4) with r instead of n_i holds clearly. Set $n_1 := r$ and let ℓ be a positive integer.

Suppose we have already defined $n_1 < \cdots < n_\ell$ such that (3.4) holds for each i with $1 \leq i \leq \ell$. Since (b_i) is greedy and $b_{n_\ell} < \alpha_1$, there exists a *smallest* integer $n_{\ell+1} > n_\ell$ such that

$$b_{n_\ell+1} \dots b_{n_{\ell+1}} < \alpha_1 \dots \alpha_{n_{\ell+1}-n_\ell}. \tag{3.5}$$

Note that $b_{n_{\ell+1}} < \alpha_{n_{\ell+1}-n_\ell} \leq \alpha_1$. It remains to verify that

$$b_{m+1} \dots b_{n_{\ell+1}} < \alpha_1 \dots \alpha_{n_{\ell+1}-m} \tag{3.6}$$

if $1 \le m < n_{\ell+1}$ and $b_m < \alpha_1$. If $m < n_\ell$, then (3.6) follows from the induction hypothesis. If $m = n_\ell$, then (3.6) reduces to (3.5). If $n_\ell < m < n_{\ell+1}$, then

$$b_{n_\ell+1} \dots b_m = \alpha_1 \dots \alpha_{m-n_\ell},$$

by minimality of $n_{\ell+1}$, and thus by (3.5) and (2.2),

$$\overline{b_{m+1} \dots b_{n_{\ell+1}}} < \alpha_{m-n_\ell+1} \dots \alpha_{n_{\ell+1}-n_\ell} \le \alpha_1 \dots \alpha_{n_{\ell+1}-m}. \qquad \square$$

We call a set $B \subset J_q$ *symmetric* if $\ell(B) = B$, where $\ell : J_q \to J_q$ is given by

$$\ell(x) = \alpha_1/(q-1) - x \qquad [x \in J_q].$$

Lemma 3.4.

(i) *The sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are symmetric.*
(ii) *The set $\mathcal{V}_q$ is closed.*

Proof. (i) The set $\mathcal{U}_q$ is symmetric because (c_i) is an expansion of x if and only if $(\alpha_1 - c_i)$ is an expansion of $\ell(x)$.

If q is a non-integer and $x \in J_q$, then by (2.4), the sequence $(\alpha_1 - a_i(x))$ is either infinite or is equal to 0^∞. It follows from Proposition 2.4 that the set $\mathcal{V}_q$ is symmetric and $(a_i(\ell(x))) = (\alpha_1 - a_i(x))$ for each $x \in \mathcal{V}_q$. If q is an integer, then $\mathcal{V}_q = J_q = [0,1]$.

(ii) We prove that $\mathcal{V}_q$ is closed by showing that its complement is open. If (a_i) is the quasi-greedy expansion of some $x \in J_q \setminus \mathcal{V}_q$, then there exists an integer $n > 0$ such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots.$$

Let m be such that

$$\overline{a_{n+1}\dots a_{n+m}} > \alpha_1 \dots \alpha_m, \tag{3.7}$$

and let

$$y = \sum_{i=1}^{n+m} \frac{a_i}{q^i}.$$

According to Lemma 3.1 the greedy expansion of y is given by $a_1 \dots a_{n+m}0^\infty$. Therefore the quasi-greedy expansion of each number $v \in (y, x]$ starts with the block $a_1 \dots a_{n+m}$. It follows from (3.7) that

$$(y, x] \cap \mathcal{V}_q = \varnothing.$$

Since $x \in J_q \setminus \mathcal{V}_q$ is arbitrary and $\mathcal{V}_q$ is symmetric, there also exists a number $z > x$ such that

$$[x, z) \cap \mathcal{V}_q = \varnothing. \qquad \square$$

Lemma 3.5. *Let (b_i) be the greedy expansion of some $x \in [0, \alpha_1/(q-1)]$ and suppose that for some $n \ge 1$,*

$$b_n > 0 \quad \textit{and} \quad \overline{b_{n+1}b_{n+2}\dots} > \alpha_1\alpha_2\dots.$$

Then

(i) *there exists a number $z > x$ such that $[x, z] \cap \mathcal{U}_q = \varnothing$;*
(ii) *if $b_j > 0$ for some $j > n$, then there exists a number $y < x$ such that $[y, x] \cap \mathcal{U}_q = \varnothing$.*

Proof. (i) Choose a positive integer $M > n$ such that

$$\overline{b_{n+1}\dots b_M} > \alpha_1 \dots \alpha_{M-n}.$$

Applying Lemma 3.2 choose a greedy sequence $(c_i) > (b_i)$ such that $c_1 \dots c_M = b_1 \dots b_M$. Then (c_i) is the greedy expansion of some $z > x$. If (d_i) is the greedy expansion of some element in $[x, z]$, then (d_i) also begins with $b_1 \dots b_M$ and hence

$$\overline{d_{n+1}\dots d_M} > \alpha_1 \dots \alpha_{M-n}.$$

In particular, we have

$$d_n > 0 \quad \text{and} \quad \overline{d_{n+1}d_{n+2}\ldots} > \alpha_1\alpha_2\ldots.$$

We infer from Theorem 1.1 that $[x,z]\cap\mathcal{U}_q = \varnothing$.

(ii) Suppose that $b_j > 0$ for some $j > n$. It follows from Lemma 3.1 that $(c_i) := b_1\ldots b_n 0^\infty$ is the greedy expansion of some $y < x$. If (d_i) is the greedy expansion of some element in $[y,x]$, then $(c_i)\le(d_i)\le(b_i)$ and $d_1\ldots d_n = b_1\ldots b_n$. Therefore

$$\overline{d_{n+1}d_{n+2}\ldots} \ge \overline{b_{n+1}b_{n+2}\ldots} > \alpha_1\alpha_2\ldots,$$

and $d_n = b_n > 0$. It follows from Theorem 1.1 that $[y,x]\cap\mathcal{U}_q = \varnothing$. □

4. Proof of Theorem 1.3

If q belongs to $\overline{\mathcal{U}}$, then we know from Theorem 1.2 and Proposition 2.3 that the quasi-greedy expansion (α_i) of 1 in base q satisfies

$$\alpha_{k+1}\alpha_{k+2}\ldots \le \alpha_1\alpha_2\ldots \quad \text{for all} \quad k\ge 1 \tag{4.1}$$

and

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} < \alpha_1\alpha_2\ldots \quad \text{for all} \quad k\ge 1. \tag{4.2}$$

Note that a sequence (α_i) satisfying (4.1) and (4.2) is automatically infinite, and is thus the quasi-greedy expansion of 1 in base q for some $q\in\overline{\mathcal{U}}$. The following lemmas are obtained in [KL3].

Lemma 4.1. *If (α_i) is a sequence satisfying (4.1) and (4.2), then there exist arbitrarily large integers m such that for all k with $0\le k<m$,*

$$\overline{\alpha_{k+1}\ldots\alpha_m} < \alpha_1\ldots\alpha_{m-k}. \tag{4.3}$$

Lemma 4.2. *Let (γ_i) be a sequence satisfying*

$$\gamma_{k+1}\gamma_{k+2}\ldots \le \gamma_1\gamma_2\ldots$$

and

$$\overline{\gamma_{k+1}\gamma_{k+2}\ldots} \le \gamma_1\gamma_2\ldots$$

for all $k\ge 1$, with $\overline{\gamma_j} := \gamma_1 - \gamma_j$, $j\ge 1$. If

$$\overline{\gamma_{n+1}\ldots\gamma_{2n}} = \gamma_1\ldots\gamma_n$$

for some $n\ge 1$, then

$$(\gamma_i) = (\gamma_1\ldots\gamma_n\overline{\gamma_1\ldots\gamma_n})^\infty.$$

Lemma 4.3. *If $q\in\overline{\mathcal{U}}\setminus\mathcal{U}$, then the greedy expansion (β_i) of 1 is finite, and all expansions of 1 are given by*

$$(\alpha_i) \quad \text{and} \quad (\alpha_1\ldots\alpha_m)^N\alpha_1\ldots\alpha_{m-1}\alpha_m^+0^\infty, \quad N=0,1,2,\ldots, \tag{4.4}$$

where m is such that β_m is the last nonzero element of (β_i).

Proof of Theorem 1.3. (i) Fix $q\in\overline{\mathcal{U}}$. It follows from Lemma 3.4 that $\overline{\mathcal{U}_q}\subset\mathcal{V}_q$. Therefore, it suffices to show that each $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ belongs to $\overline{\mathcal{U}_q}$.

First assume that $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ has a finite greedy expansion (b_i). If b_n is the last nonzero element of (b_i), then

$$(a_i(x)) = (a_i) = b_1\ldots b_n^-\alpha_1\alpha_2\ldots.$$

According to Lemma 4.1 there exists a sequence $1\le m_1<m_2<\cdots$ such that (4.3) is satisfied with $m=m_i$ for all $i\ge 1$. We may assume that $m_i>n$ for all $i\ge 1$. Consider for each $i\ge 1$ the sequence (b_j^i), given by

$$(b_j^i) = b_1\ldots b_n^-(\alpha_1\ldots\alpha_{m_i}\overline{\alpha_1\ldots\alpha_{m_i}})^\infty,$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

Note that the sequence $(x_i)_{i\geq 1}$ converges to x as i goes to infinity. It remains to show that $x_i \in \mathcal{U}_q$ for all $i \geq 1$. According to Theorem 1.1 it suffices to verify that

$$b_{m+1}^i b_{m+2}^i \ldots < \alpha_1\alpha_2 \ldots \quad \text{whenever} \quad b_m^i < \alpha_1 \tag{4.5}$$

and

$$\overline{b_{m+1}^i b_{m+2}^i \ldots} < \alpha_1\alpha_2 \ldots \quad \text{whenever} \quad b_m^i > 0. \tag{4.6}$$

According to (4.2) we have

$$\overline{\alpha_{m_i+1} \ldots \alpha_{2m_i}} \leq \alpha_1 \ldots \alpha_{m_i}.$$

Note that this inequality cannot be an equality, for otherwise it would follow from Lemma 4.2 that

$$(\alpha_i) = (\alpha_1 \ldots \alpha_{m_i}\overline{\alpha_1 \ldots \alpha_{m_i}})^{\infty}.$$

However, this sequence does not satisfy (4.2) for $k = m_i$. Therefore

$$\overline{\alpha_{m_i+1} \ldots \alpha_{2m_i}} < \alpha_1 \ldots \alpha_{m_i}$$

or equivalently

$$\overline{\alpha_1 \ldots \alpha_{m_i}} < \alpha_{m_i+1} \ldots \alpha_{2m_i}. \tag{4.7}$$

If $m \geq n$, then (4.5) and (4.6) follow from (4.1), (4.3) and (4.7). Now we verify (4.5) and (4.6) for $m < n$. Fix $m < n$. If $b_m^i < \alpha_1$, then

$$b_{m+1}^i \ldots b_n^i = b_{m+1} \ldots b_n^- < b_{m+1} \ldots b_n \leq \alpha_1 \ldots \alpha_{n-m},$$

where the last inequality follows from the fact that (b_i) is a greedy expansion and $b_m = b_m^i < \alpha_1$. Hence

$$b_{m+1}^i b_{m+2}^i \ldots < \alpha_1\alpha_2 \ldots.$$

Suppose that $b_m^i = a_m > 0$. Since

$$\overline{a_{m+1}a_{m+2} \ldots} \leq \alpha_1\alpha_2 \ldots$$

by assumption, and $b_{m+1}^i \ldots b_n^i = a_{m+1} \ldots a_n$, it suffices to verify that

$$\overline{b_{n+1}^i b_{n+2}^i \ldots} < \alpha_{n-m+1}\alpha_{n-m+2} \ldots.$$

This is equivalent to

$$\overline{\alpha_{n-m+1}\alpha_{n-m+2} \ldots} < (\alpha_1 \ldots \alpha_{m_i}\overline{\alpha_1 \ldots \alpha_{m_i}})^{\infty}. \tag{4.8}$$

Since $n < m_i$ for all $i \geq 1$, we infer from (4.3) that

$$\overline{\alpha_{n-m+1} \ldots \alpha_{m_i}} < \alpha_1 \ldots \alpha_{m_i-(n-m)},$$

and (4.8) follows.

Next assume that $x \in \mathcal{V}_q \backslash \mathcal{U}_q$ has an infinite greedy expansion (b_i). Since $x \notin \mathcal{U}_q$, there exists a *smallest* positive integer n such that

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1}b_{n+2} \ldots} \geq \alpha_1\alpha_2 \ldots. \tag{4.9}$$

Since $x \in \mathcal{V}_q$ and $(a_i(x)) = (b_i)$, this last inequality is in fact an equality. As before, let $1 \leq m_1 < m_2 < \cdots$ be a sequence such that (4.3) is satisfied with $m = m_i$ for all $i \geq 1$. Again, we may assume that $m_i > n$ for all $i \geq 1$. Consider for each $i \geq 1$ the sequence (b_j^i), given by

$$(b_j^i) = b_1 \ldots b_n(\overline{\alpha_1 \ldots \alpha_{m_i}}\alpha_1 \ldots \alpha_{m_i})^{\infty},$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

Then the sequence $(x_i)_{i\geq 1}$ converges to x as i goes to infinity. It remains to show that $x_i \in \mathcal{U}_q$ for all $i \geq 1$, i.e., it remains to verify (4.5) and (4.6). We leave this easy verification to the reader.

(iia) We establish that $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ for each $q \in \overline{\mathcal{U}}$. More specifically, if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then the sets A_q and B_q (introduced in a remark following the statement of Theorem 1.3) are countably infinite, and the greedy expansion of a number $x \in B_q$ ends with $\overline{\alpha_1\alpha_2\ldots}$. If $q \in \{2, 3, \ldots\}$, then $A_q = \mathcal{V}_q \setminus \mathcal{U}_q$.

Fix $q \in \overline{\mathcal{U}}$. Denote the greedy expansion of a number $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ by (b_i). Since $x \notin \mathcal{U}_q$, there exists a number n such that (4.9) holds. If both inequalities in (4.9) are strict, then $b_i = 0$ for all $i > n$, as follows from Lemma 3.5 and part (i). Otherwise, the sequence (b_i) ends with $\overline{\alpha_1\alpha_2\ldots}$, which is infinite unless q is an integer. It follows from Theorems 1.1 and 1.2 that a sequence of the form $0^n 10^\infty$ $(n \geq 0)$ is the finite greedy expansion of $1/q^{n+1} \in \mathcal{V}_q \setminus \mathcal{U}_q$. Moreover, if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then a sequence of the form $\alpha_1^n \overline{\alpha_1\alpha_2\ldots}$ $(n \geq 1)$ is the infinite greedy expansion of a number $x \in \mathcal{V}_q \setminus \mathcal{U}_q$. These observations conclude the proof.

(iib) We show that if $q \in \overline{\mathcal{U}}$, then A_q is dense in $\mathcal{V}_q$.

Fix $q \in \overline{\mathcal{U}}$. For each $x \in \mathcal{U}_q$, we will define a sequence $(x_i)_{i\geq 1}$ of numbers in $A_q \subset \mathcal{V}_q \setminus \mathcal{U}_q$ that converges to x. We have seen in the proof of part (iia) that $1/q^n \in A_q$ for each $n \geq 1$. Hence there exists a sequence of numbers in A_q that converges to 0. Now suppose that $x \in \mathcal{U}_q \setminus \{0\}$ and denote by (c_i) the unique expansion of x. Since $\overline{c_1c_2\ldots} \neq \alpha_1^\infty$ is greedy, we infer from Lemma 3.3 that there exists a sequence $1 \leq n_1 < n_2 < \cdots$, such that for each $i \geq 1$,

$$c_{n_i} > 0 \quad \text{and} \quad \overline{c_{m+1}\ldots c_{n_i}} < \alpha_1 \ldots \alpha_{n_i - m} \quad \text{if } m < n_i \text{ and } c_m > 0. \tag{4.10}$$

Now consider for each $i \geq 1$ the sequence (b_j^i), given by

$$(b_j^i) = c_1 \ldots c_{n_i} 0^\infty,$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

According to Lemma 3.1 the sequence (b_j^i) is the finite greedy expansion of the number x_i, $i \geq 1$. Moreover, the sequence $(x_i)_{i\geq 1}$ converges to x as i goes to infinity. We claim that $x_i \in A_q$ for each $i \geq 1$. Note that $x_i \notin \mathcal{U}_q$ because the quasi-greedy sequence (a_j^i), given by

$$c_1 \ldots c_{n_i}^- \alpha_1 \alpha_2 \ldots,$$

is another expansion of x_i. It remains to prove that

$$a_j^i > 0 \Longrightarrow \overline{a_{j+1}^i a_{j+2}^i \ldots} \leq \alpha_1 \alpha_2 \ldots. \tag{4.11}$$

If $j < n_i$ and $a_j^i > 0$, then

$$\overline{a_{j+1}^i \ldots a_{n_i}^i} = \overline{c_{j+1} \ldots c_{n_i}^-} \leq \alpha_1 \ldots \alpha_{n_i - j}$$

by (4.10), and

$$\overline{a_{n_i+1}^i a_{n_i+2}^i \ldots} = \overline{\alpha_1 \alpha_2 \ldots} < \alpha_{n_i-j+1} \alpha_{n_i-j+2} \ldots$$

by Theorem 1.2. If $j = n_i$, then (4.11) follows from $\overline{\alpha_1} = 0 < \alpha_1$. Finally, if $j > n_i$, then (4.11) follows again from Theorem 1.2.

(iic) We show that if $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then the set B_q is dense in $\mathcal{V}_q$, and

$$B_q = \{\alpha_1/(q-1) - x : x \in A_q\}.$$

Fix $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$ and suppose that $x \in A_q$ has a finite greedy expansion (b_i) with last nonzero element b_n. An application of Proposition 2.5 and Theorem 1.2 yields that

$$(c_i) = \overline{b_1 \dots b_n^- \alpha_1 \alpha_2 \dots}$$

is the greedy expansion of $\alpha_1/(q-1) - x$. It follows from the symmetry of $\mathcal{U}_q$ and $\mathcal{V}_q$ that the number $\alpha_1/(q-1) - x$ belongs to B_q. Conversely, suppose that $x \in B_q$ has an infinite greedy expansion (b_i) and let n be the *smallest* positive integer for which (4.9) holds. Then $b_{n+1}b_{n+2}\dots = \overline{\alpha_1\alpha_2\dots}$, and

$$(c_i) = \overline{b_1 \dots b_n^-} 0^\infty$$

is the greedy expansion of $\alpha_1/(q-1) - x \in A_q$. It follows from the symmetry of $\mathcal{U}_q$ together with part (iib) that the set B_q is dense in $\mathcal{V}_q$ as well.

(iii) and (iv) Fix $q \in \overline{\mathcal{U}}$ and let (b_i) be the greedy expansion of a number $x \in \mathcal{V}_q \setminus \mathcal{U}_q$. Let n be the *smallest* positive integer for which (4.9) holds and let (d_i) be another expansion of x. Then $(d_i) < (b_i)$, and hence there exists a *smallest* integer $j \ge 1$ for which $d_j < b_j$. First we show that $j \ge n$. Assume on the contrary that $j < n$. Then $b_j > 0$, and by minimality of n we have

$$b_{j+1}b_{j+2}\dots > \overline{\alpha_1\alpha_2\dots}.$$

From Theorems 1.1 and 1.2 we know that $\overline{\alpha_1\alpha_2\dots}$ is the greedy expansion of $\alpha_1/(q-1) - 1$, and thus

$$\sum_{i=1}^\infty \frac{d_{j+i}}{q^i} = b_j - d_j + \sum_{i=1}^\infty \frac{b_{j+i}}{q^i} > 1 + \frac{\alpha_1}{q-1} - 1 = \frac{\alpha_1}{q-1},$$

which is impossible. If $j = n$, then $d_n = b_n^-$, for otherwise we have $q > 2$ and

$$2 \le \sum_{i=1}^\infty \frac{d_{n+i}}{q^i} \le \frac{\lceil q \rceil - 1}{q-1},$$

which is also impossible. Now we distinguish between two cases.

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\dots} > \alpha_1\alpha_2\dots, \tag{4.12}$$

then by Lemma 3.5 and part (i) we have $b_r = 0$ for all $r > n$, from which it follows that (d_{n+i}) is an expansion of 1. Hence, if $q \in \mathcal{U}$ and (4.12) holds, then the only expansion of x starting with $b_1 \dots b_n^-$ is given by $(c_i) := b_1 \dots b_n^- \alpha_1\alpha_2\dots$. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$ and (4.12) holds, then any expansion (c_i) starting with $b_1 \dots b_n^-$ is an expansion of x if and only if (c_{n+i}) is one of the expansions listed in (4.4).

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\dots} = \alpha_1\alpha_2\dots, \tag{4.13}$$

then

$$\sum_{i=1}^\infty \frac{d_{n+i}}{q^i} = 1 + \sum_{i=1}^\infty \frac{b_{n+i}}{q^i} = \sum_{i=1}^\infty \frac{\alpha_1}{q^i}.$$

Hence, if (4.13) holds, then the only expansion of x starting with $b_1 \dots b_n^-$ is given by $b_1 \dots b_n^- \alpha_1^\infty$.

Finally, if $j > n$, then (4.13) holds, for otherwise $(b_{n+i}) = 0^\infty$ and $d_j < b_j$ is impossible. Note that in this case $q \notin \mathcal{U}$, because otherwise (b_{n+i}) is the unique expansion of $\sum_{i=1}^\infty \overline{\alpha_i} q^{-i}$ and thus $(d_{n+i}) = (b_{n+i})$ which is impossible due to $j > n$. Hence, if $q \in \mathcal{U}$, then (b_i) is the only expansion of x starting with $b_1 \dots b_n$. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$ and (4.13) holds, then any expansion (c_i) starting with $b_1 \dots b_n$ is an

expansion of x if and only if (c_{n+i}) is one of the conjugates of the expansions listed in (4.4).

Parts (iii) and (iv) follow directly from the above considerations. $\square$

Remarks.

- Fix $q \in \overline{\mathcal{U}}$. It follows from the proof of Theorem 1.3 (iia) that each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has either a finite expansion or an expansion that ends with $\overline{\alpha_1\alpha_2\ldots}$ in which case x can be written as
$$x = \frac{b_1}{q} + \cdots + \frac{b_n}{q^n} + \frac{1}{q^n}\left(\frac{\alpha_1}{q-1} - 1\right).$$
Hence, if q is algebraic, then each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ is algebraic, and if q is transcendental, then each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ is transcendental. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then q is algebraic because 1 has a finite greedy expansion in base q by Lemma 4.3. If $q \in \mathcal{U}$, then each neighborhood of q contains uncountably many univoque numbers because $\overline{\mathcal{U}}$ is a perfect set and $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable ([KL3]). Hence the set of transcendental univoque numbers is dense in $\overline{\mathcal{U}}$. For instance, it was shown by Allouche and Cosnard in [AC] that the smallest univoque number q' is transcendental. Subsequently, it was shown in [DV1] that the set of algebraic univoque numbers is dense in $\overline{\mathcal{U}}$ as well. This implies in particular that there does not exist a smallest algebraic univoque number, a result first established in [KLPt].
- It follows from Theorem 1.3 (ii) and (iii) that the set
$$\mathcal{T}_q := \{x \in J_q : x \text{ has exactly 2 expansions in base } q\}$$
is not closed if q belongs to $\mathcal{U}$, in which case its closure contains $\mathcal{U}_q$. It would be interesting to determine all numbers $q > 1$ for which $\mathcal{T}_q$ is not closed.

5. Proof of Theorems 1.4 and 1.5

Fix $q > 1$. It follows from Propositions 2.3 and 2.5 that a sequence (b_i) is greedy if and only if $0 \le b_n \le \alpha_1$ for all $n \ge 1$, and

$$b_{n+k+1}b_{n+k+2}\ldots < \alpha_1\alpha_2\ldots \quad \text{for all } k \ge 0, \quad \text{whenever} \quad b_n < \alpha_1. \tag{5.1}$$

Lemma 5.1. *Suppose that $q \notin \overline{\mathcal{U}}$. Then a greedy sequence (b_i) cannot end with $\overline{\alpha_1\alpha_2\ldots}$.*

Proof. Assume on the contrary that for some $n \ge 0$,
$$b_{n+1}b_{n+2}\ldots = \overline{\alpha_1\alpha_2\ldots}.$$
Since in this case $b_{n+1} = \overline{\alpha_1} = 0 < \alpha_1$, it would follow from (5.1) that
$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} < \alpha_1\alpha_2\ldots \quad \text{for all} \quad k \ge 1.$$
But this contradicts Theorem 1.2. $\square$

Lemma 5.2. *Suppose that $q \notin \overline{\mathcal{U}}$. Then*

(i) *the set $\mathcal{U}_q$ is closed;*
(ii) *each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion.*

Proof. (i) Let $x \in J_q \setminus \mathcal{U}_q$ and denote the greedy expansion of x in base q by (b_i). According to Theorem 1.1 there exists a positive integer n such that
$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1}b_{n+2}\ldots} \ge \alpha_1\alpha_2\ldots.$$
Applying Lemmas 3.5 and 5.1 we conclude that
$$[x, z] \cap \mathcal{U}_q = \varnothing$$

for some number $z > x$. It follows that $\mathcal{U}_q$ is closed from above. Since the set $\mathcal{U}_q$ is symmetric it is closed from below as well.

(ii) Assume on the contrary that $(a_i(x)) = (b_i(x))$ for some $x \in \mathcal{V}_q \setminus \mathcal{U}_q$. Then it would follow that for some positive integer n,

$$\overline{b_{n+1}(x)b_{n+2}(x)\ldots} = \alpha_1\alpha_2\ldots,$$

contradicting Lemma 5.1. □

Recall from the introduction that the set $\mathcal{V}$ consists of those numbers $q > 1$ for which the quasi-greedy expansion (α_i) of 1 in base q satisfies

$$\overline{\alpha_{n+1}\alpha_{n+2}\ldots} \le \alpha_1\alpha_2\ldots \quad \text{for all} \quad n \ge 1. \tag{5.2}$$

If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then the quasi-greedy expansion of 1 in base q is of the form

$$(\alpha_i) = (\alpha_1\ldots\alpha_k\overline{\alpha_1\ldots\alpha_k})^\infty, \tag{5.3}$$

where k is the least positive integer satisfying

$$\overline{\alpha_{k+1}\alpha_{k+2}\ldots} = \alpha_1\alpha_2\ldots. \tag{5.4}$$

In particular, such a sequence is periodic. Note that $\alpha_k > 0$, for otherwise it would follow from (5.2) and (5.3) that

$$\overline{\alpha_k\alpha_{k+1}\ldots\alpha_{2k-1}} = \alpha_1(\alpha_1\ldots\alpha_{k-1}) \le \alpha_1\ldots\alpha_{k-1}0,$$

which is impossible because $\alpha_1 > 0$ and $\alpha_j \le \alpha_1$ for each $j \in \mathbb{N}$. Any sequence of the form $(1^m0^m)^\infty$, where m is a positive integer, is infinite and satisfies (4.1) and (5.2) but not (4.2). On the other hand, there are only countably many periodic sequences. Hence the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ is countably infinite.

The following lemma ([KL3]) implies that the number of expansions of 1 is countably infinite in case $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Moreover, all expansions of the number 1 in such a base q are determined explicitly.

Lemma 5.3. *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then all expansions of 1 are given by (α_i), and the sequences*

$$(\alpha_1\ldots\alpha_{2k})^N\alpha_1\ldots\alpha_{2k-1}\alpha_{2k}^+0^\infty\,, \quad N = 0, 1, \ldots$$

and

$$(\alpha_1\ldots\alpha_{2k})^N\alpha_1\ldots\alpha_{k-1}\alpha_k^-\alpha_1^\infty\,, \quad N = 0, 1, \ldots.$$

Now we are ready to prove Theorems 1.4 and 1.5. Throughout the proof of Theorem 1.4, $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ is fixed but arbitrary, and k is the least positive integer satisfying (5.4) with $(\alpha_i) = (\alpha_i(q))$.

Proof of Theorem 1.4. Thanks to Lemmas 3.4 and 5.2 we only need to prove parts (ii) and (iii).

(iia) We prove that $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$. The set $\mathcal{V}_q \setminus \mathcal{U}_q$ is countable because each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion (see Lemma 5.2). On the other hand, for each $n \ge 1$ the sequence $\alpha_1^n0^\infty$ is the greedy expansion of an element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$, from which the claim follows.

(iib) In order to show that $\mathcal{V}_q \setminus \mathcal{U}_q$ is dense in $\mathcal{V}_q$, one can argue as in the proof of Theorem 1.3 (iib). Instead of applying Theorem 1.2 one should now apply the inequalities (5.2).

(iic) Finally, we show that all elements of $\mathcal{V}_q \setminus \mathcal{U}_q$ are isolated points of $\mathcal{V}_q$. Let $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ and let b_n be the last nonzero element of the greedy expansion (b_i) of x. Choose m such that $\alpha_m < \alpha_1$. This is possible because $q \notin \mathbb{N}$. According to Lemma 3.2 there exists a greedy sequence $(c_i) > (b_i)$ such that

$$c_1\ldots c_{n+m} = b_1\ldots b_n0^m.$$

If we set

$$z=\sum_{i=1}^{\infty}\frac{c_i}{q^i},$$

then the quasi-greedy expansion (v_i) of a number $v\in(x,z]$ starts with $b_1\ldots b_n0^m$. Hence $v_n=b_n>0$ and

$$\overline{v_{n+1}\ldots v_{n+m}}=\alpha_1^m>\alpha_1\ldots\alpha_m.$$

Therefore

$$(x,z]\cap\mathcal{V}_q=\varnothing.$$

Since the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are symmetric, there also exists a number $y<x$ satisfying

$$(y,x)\cap\mathcal{V}_q=\varnothing.$$

(iii) We already know from Lemma 5.2 that each $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ has a finite greedy expansion. It remains to show that each element $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ has exactly $\aleph_0$ expansions. Let $x\in\mathcal{V}_q\setminus\mathcal{U}_q$ and let b_n be the last nonzero element of its greedy expansion (b_i). If $j<n$ and $b_j=a_j(x)>0$, then

$$\overline{a_{j+1}(x)\ldots a_n(x)}=\overline{b_{j+1}\ldots b_n^-}\le\alpha_1\ldots\alpha_{n-j},$$

because $x\in\mathcal{V}_q$. Therefore

$$b_{j+1}\ldots b_n>\overline{\alpha_1\ldots\alpha_{n-j}}. \tag{5.5}$$

Let (d_i) be another expansion of x and let j be the *smallest* positive integer for which $d_j\ne b_j$. Since (b_i) is greedy, we have $d_j<b_j$ and $j\in\{1,\ldots,n\}$. First we show that $j\in\{n-k,n\}$. Assume on the contrary that $j\notin\{n-k,n\}$.

First assume that $n-k<j<n$. Then $b_j>0$ and by (5.5),

$$b_{j+1}\ldots b_n0^\infty>\overline{\alpha_1\ldots\alpha_{n-j}}\,\overline{\alpha_{n-j+1}\ldots\alpha_k^-}0^\infty.$$

Since $\alpha_1\ldots\alpha_k^-\alpha_1^\infty$ is the smallest expansion of 1 in base q (see Lemma 5.3), $\overline{\alpha_1\ldots\alpha_k^-}0^\infty$ is the greedy expansion of $\alpha_1/(q-1)-1$, and thus

$$\sum_{i=1}^{\infty}\frac{d_{j+i}}{q^i}=b_j-d_j+\sum_{i=1}^{\infty}\frac{b_{j+i}}{q^i}>\frac{\alpha_1}{q-1},$$

which is impossible.

Next assume that $1\le j<n-k$. Rewriting (5.5) one gets

$$\overline{b_{j+1}\ldots b_n}<\alpha_1\ldots\alpha_{n-j}.$$

If we had

$$\overline{b_{j+1}\ldots b_{j+k}}=\alpha_1\ldots\alpha_k,$$

then

$$\overline{b_{j+k+1}\ldots b_n}<\alpha_{k+1}\ldots\alpha_{n-j}.$$

Hence

$$b_{j+k+1}b_{j+k+2}\ldots>\overline{\alpha_{k+1}\alpha_{k+2}\ldots}=\alpha_1\alpha_2\ldots.$$

Since in this case $b_{j+k}=\overline{\alpha_k}<\alpha_1$, the last inequality contradicts the fact that (b_i) is a greedy sequence. Therefore

$$\overline{b_{j+1}\ldots b_{j+k}}<\alpha_1\ldots\alpha_k$$

or equivalently

$$b_{j+1}\ldots b_{j+k}\ge\overline{\alpha_1\ldots\alpha_k^-}.$$

Since $n>j+k$ and $b_n>0$, it follows that

$$b_{j+1}b_{j+2}\ldots>\overline{\alpha_1\ldots\alpha_k^-}0^\infty,$$

which leads to the same contradiction as we encountered above. It remains to investigate what happens if $j \in \{n-k, n\}$.

If $j = n-k$, then it follows from (5.5) that

$$b_{n-k+1} \ldots b_n \geq \overline{\alpha_1 \ldots \alpha_k^-}.$$

Equivalently,

$$b_{n-k+1} b_{n-k+2} \ldots = b_{n-k+1} \ldots b_n 0^\infty \geq \overline{\alpha_1 \ldots \alpha_k^-} 0^\infty,$$

and thus

$$\sum_{i=1}^{\infty} \frac{d_{n-k+i}}{q^i} \geq 1 + \sum_{i=1}^{\infty} \frac{b_{n-k+i}}{q^i} \geq \frac{\alpha_1}{q-1}, \tag{5.6}$$

where both inequalities in (5.6) are equalities if and only if

$$d_{n-k} = b_{n-k}^-,\ b_{n-k+1} \ldots b_n = \overline{\alpha_1 \ldots \alpha_k^-}, \text{ and } d_{n-k+1} d_{n-k+2} \ldots = \alpha_1^\infty.$$

Hence $d_{n-k} < b_{n-k}$ is only possible in case $b_{n-k} > 0$ and $b_{n-k+1} \ldots b_n = \overline{\alpha_1 \ldots \alpha_k^-}$.

Finally, if $j = n$, then $d_n = b_n^-$ and (d_{n+i}) is one of the expansions listed in Lemma 5.3. □

Remark. Fix $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. By Lemma 5.3 the number 1 has a finite greedy expansion in base q. Hence each element $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ is algebraic. Since each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion in base q, it follows that the set $\mathcal{V}_q \setminus \mathcal{U}_q$ consists entirely of algebraic numbers.

Proof of Theorem 1.5. Fix $q \in (1, \infty) \setminus \mathcal{V}$. In view of Lemma 5.2 it suffices to prove that a number $x \in J_q \setminus \{0\}$ with a finite greedy expansion does *not* belong to $\mathcal{V}_q$.

Let $x \in J_q \setminus \{0\}$ be an element with a finite greedy expansion. Since $q \notin \mathcal{V}$, there exists a positive integer n such that

$$\overline{\alpha_{n+1} \alpha_{n+2} \ldots} > \alpha_1 \alpha_2 \ldots.$$

Let $m = \max\{i \in \mathbb{N} : 1 \leq i \leq n \text{ and } \alpha_i > 0\}$. From Proposition 2.3 it follows at once that

$$\alpha_m > 0 \quad \text{and} \quad \overline{\alpha_{m+1} \alpha_{m+2} \ldots} > \alpha_1 \alpha_2 \ldots.$$

Since $(a_i(x))$ ends with $\alpha_1 \alpha_2 \ldots$, we conclude that $x \notin \mathcal{V}_q$. □

Corollary 5.4. *A real number $q > 1$ belongs to $\overline{\mathcal{U}}$ if and only if 1 has a unique* infinite *expansion in base q.*

Proof. Suppose first that $q \in \overline{\mathcal{U}}$. It follows from Theorems 1.1 and 1.2 that $(\alpha_i(q))$ is the smallest expansion of 1 in base q. Since $(\alpha_i(q))$ is by definition its largest infinite expansion in base q, it follows that $(\alpha_i(q))$ is the unique infinite expansion of 1 in base q. Conversely, if 1 has a unique infinite expansion in base q, then $(\alpha_i(q)) = (\alpha_i)$ is its smallest expansion because the smallest expansion of 1 is infinite in any base $q > 1$. Hence $\overline{\alpha_1 \alpha_2 \ldots}$ is a greedy sequence in base q. Applying (5.1) (note that $\overline{\alpha_1} = 0$) and Theorem 1.2 we conclude that $q \in \overline{\mathcal{U}}$. □

6. Proof of Theorems 1.6, 1.7, 1.8 and 1.9

In this section we will complete our study of the sets $\mathcal{U}_q$ for numbers $q > 1$. The results proved in the preceding sections were mainly concerned with various properties of these sets for numbers $q \in \mathcal{V}$. Now we will use these properties to describe the topological structure of $\mathcal{U}_q$ for each number $q > 1$.

Since the set $\mathcal{V}$ is closed, we may write $(1, \infty) \setminus \mathcal{V}$ as the union of countably many disjoint open intervals (q_1, q_2): the connected components of $(1, \infty) \setminus \mathcal{V}$. In order to determine the endpoints of these components we recall from [KL3] that $\mathcal{V} \setminus \overline{\mathcal{U}}$

is dense in $\mathcal{V}$ and all elements of $\mathcal{V} \setminus \overline{\mathcal{U}}$ are isolated in $\mathcal{V}$. In fact, for each element $q \in \overline{\mathcal{U}}$ there exists a sequence $(q_m)_{m\geq 1}$ of numbers in $\mathcal{V} \setminus \overline{\mathcal{U}}$ such that $q_m \uparrow q$, as can be seen from the proof of Theorem 2.6 in [KL3].

Proposition 6.1.

(i) *The set R of right endpoints q_2 of the connected components (q_1, q_2) is given by $R = \mathcal{V} \setminus \overline{\mathcal{U}}$.*

(ii) *The set L of left endpoints q_1 of the connected components (q_1, q_2) is given by $L = \mathbb{N} \cup (\mathcal{V} \setminus \mathcal{U})$.*

Proof of Proposition 6.1 (i). Note that $\mathcal{V} \setminus \overline{\mathcal{U}} \subset R$ because the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ is discrete. As we have already observed above, each element $q \in \overline{\mathcal{U}}$ can be approximated arbitrarily closely by elements of $\mathcal{V}$ smaller than q, and thus $R = \mathcal{V} \setminus \overline{\mathcal{U}}$. $\square$

The proof of part (ii) of Proposition 6.1 requires more work. We will first prove a number of technical lemmas. We recall from Section 1 that the notation $q \sim (\alpha_i)$ means that the quasi-greedy expansion of 1 in base q is given by (α_i). For convenience we also write $1 \sim 0^\infty$, and occasionally we refer to 0^∞ as the quasi-greedy expansion of the number 1 in base 1.

In Lemmas 6.2 and 6.3 below, q_2 is a fixed (but arbitrary) element of $\mathcal{V} \setminus \overline{\mathcal{U}}$, and

$$(\alpha_i) = (\alpha_i(q_2)) = (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty$$

where k is chosen to be minimal.

Remark. The minimality of k implies that the least period of (α_i) equals $2k$. Indeed, if j is the least period of (α_i), then $\alpha_j = \alpha_{2k} = \overline{\alpha_k} < \alpha_1$ because j divides $2k$. Hence $\alpha_1 \ldots \alpha_j^+ 0^\infty$ is an expansion of 1 in base q_2 which contradicts Lemma 5.3 if $j < 2k$.

Lemma 6.2. *For all i with $0 \leq i < k$, we have*

$$\overline{\alpha_{i+1} \ldots \alpha_k} < \alpha_1 \ldots \alpha_{k-i}.$$

Proof. For $i = 0$ the inequality follows from the relation $\overline{\alpha_1} = 0 < \alpha_1$. Henceforth assume that $1 \leq i < k$. Since $q_2 \in \mathcal{V}$,

$$\overline{\alpha_{i+1} \ldots \alpha_k} \leq \alpha_1 \ldots \alpha_{k-i}.$$

Suppose that

$$\overline{\alpha_{i+1} \ldots \alpha_k} = \alpha_1 \ldots \alpha_{k-i}.$$

If $k \geq 2i$, then

$$\alpha_1 \ldots \alpha_{2i} = \alpha_1 \ldots \alpha_i \overline{\alpha_1 \ldots \alpha_i},$$

and it would follow from Lemma 4.2 that

$$(\alpha_i) = (\alpha_1 \ldots \alpha_i \overline{\alpha_1 \ldots \alpha_i})^\infty,$$

contradicting the minimality of k. If $i < k < 2i$, then

$$\begin{aligned} \overline{\alpha_{i+1} \ldots \alpha_{2i}} &= \overline{\alpha_{i+1} \ldots \alpha_k} \alpha_1 \ldots \alpha_{2i-k} \\ &= \alpha_1 \ldots \alpha_{k-i} \alpha_1 \ldots \alpha_{2i-k} \\ &\geq \alpha_1 \ldots \alpha_{k-i} \alpha_{k-i+1} \ldots \alpha_i \\ &= \alpha_1 \ldots \alpha_i, \end{aligned}$$

leading to the same contradiction. $\square$

Let q_1 be the largest element of $\mathcal{V} \cup \{1\}$ that is smaller than q_2. This element exists because the set $\mathcal{V} \cup \{1\}$ is closed and the elements of $\mathcal{V} \setminus \overline{\mathcal{U}}$ are isolated points of $\mathcal{V} \cup \{1\}$. The next lemma provides the quasi-greedy expansion of 1 in base q_1.

Lemma 6.3. $q_1 \sim (\alpha_1 \ldots \alpha_k^-)^\infty$.

Proof. Let $q_1 \sim (v_i)$. If $k = 1$, then $q_2 \sim (\alpha_1 0)^\infty$, and $(v_i) = (\alpha_1^-)^\infty$ because q_2 is the smallest element of $\mathcal{V} \cap (\alpha_1, \alpha_1 + 1)$. Hence we may assume that $k \geq 2$. This implies in particular that $q_1 \in \mathcal{V}$ and $\lceil q_1 \rceil = \lceil q_2 \rceil$. Observe that

$$v_1 \dots v_k \leq \alpha_1 \dots \alpha_k.$$

If we had

$$v_1 \dots v_k = \alpha_1 \dots \alpha_k,$$

then

$$v_{k+1} \dots v_{2k} \leq \overline{\alpha_1 \dots \alpha_k},$$

i.e.,

$$\overline{v_{k+1} \dots v_{2k}} \geq \alpha_1 \dots \alpha_k = v_1 \dots v_k$$

and it would follow from Lemma 4.2 that $q_1 = q_2$. Hence

$$v_1 \dots v_k \leq \alpha_1 \dots \alpha_k^-.$$

It follows from Proposition 2.3 that $(w_i) = (\alpha_1 \dots \alpha_k^-)^\infty$ is the largest quasi-greedy expansion of 1 in some base $q > 1$ that starts with $\alpha_1 \dots \alpha_k^-$. Therefore it suffices to show that the sequence (w_i) satisfies the inequalities (5.2). Since the sequence (w_i) is periodic with period k, it is sufficient to verify that

$$\overline{w_{j+1} w_{j+2} \dots} \leq w_1 w_2 \dots \tag{6.1}$$

for all j with $0 \leq j < k$. If $j = 0$, then (6.1) is true because $\overline{w_1} = 0 < w_1$; hence assume that $1 \leq j < k$. Then, according to the preceding lemma,

$$\overline{\alpha_{j+1} \dots \alpha_k} < \alpha_1 \dots \alpha_{k-j}$$

and

$$\overline{\alpha_1 \dots \alpha_j} < \alpha_{k-j+1} \dots \alpha_k.$$

Hence

$$\begin{aligned} \overline{w_{j+1} \dots w_{j+k}} &= \overline{\alpha_{j+1} \dots \alpha_k^-}\,\overline{\alpha_1 \dots \alpha_j} \\ &\leq \alpha_1 \dots \alpha_{k-j} \overline{\alpha_1 \dots \alpha_j} \\ &< \alpha_1 \dots \alpha_k, \end{aligned}$$

so that

$$\overline{w_{j+1} \dots w_{j+k}} \leq w_1 \dots w_k.$$

Since the sequence $(w_{j+i}) = w_{j+1} w_{j+2} \dots$ is also periodic with period k, the inequality (6.1) follows. $\square$

We include for completeness the following lemma (see also [KL3]).

Lemma 6.4. *Fix $q > 1$ and let $(\beta_i) = (b_i(1, q))$ be the greedy expansion of the number 1 in base q. For any positive integer n, we have*

$$\beta_{n+1} \beta_{n+2} \dots \leq \beta_1 \beta_2 \dots.$$

Proof. Let $n \in \mathbb{N}$. From (5.1) we get that

$$\beta_{n+1} \beta_{n+2} \dots < \alpha_1 \alpha_2 \dots \leq \beta_1 \beta_2 \dots,$$

whenever there exists a positive integer $j \leq n$ satisfying $\beta_j < \beta_1 = \alpha_1$. If such an integer j does not exist, then either $(\beta_i) = \alpha_1^\infty$ or there exists an integer $j > n$ for which $\beta_j < \alpha_1$. In both these cases the desired inequality readily follows as well. $\square$

Now we consider a number $q_1 \in \mathcal{V} \setminus \mathcal{U}$. Recall from Lemmas 4.3 and 5.3 that the greedy expansion (β_i) of 1 in base q_1 is finite. Let β_m be its last nonzero element.

Lemma 6.5.

(i) *The least element* q_2 *of* $\mathcal{V}$ *that is larger than* q_1 *exists. Moreover,*

$$q_2 \sim (\beta_1 \dots \beta_m \overline{\beta_1 \dots \beta_m})^\infty.$$

(ii) *The greedy expansion of* 1 *in base* q_2 *is given by* $(\gamma_i) = \beta_1 \dots \beta_m \overline{\beta_1 \dots \beta_m^-} 0^\infty$.

Proof. (i) First of all, note that

$$q_1 \sim (\alpha_i) = (\beta_1 \dots \beta_m^-)^\infty.$$

Moreover, $(\beta_1 \dots \beta_m^-)^\infty$ is the largest quasi-greedy expansion of 1 in some base $q > 1$ that starts with $\beta_1 \dots \beta_m^-$. Hence, in view of Lemma 4.2, it suffices to show that the infinite sequence

$$(w_i) = (\beta_1 \dots \beta_m \overline{\beta_1 \dots \beta_m})^\infty$$

satisfies the inequalities

$$w_{k+1} w_{k+2} \dots \le w_1 w_2 \dots \tag{6.2}$$

and

$$\overline{w_{k+1} w_{k+2} \dots} \le w_1 w_2 \dots \tag{6.3}$$

for all $k \ge 0$. Observe that (6.2) for $k+m$ is equivalent to (6.3) for k and (6.3) for $k+m$ is equivalent to (6.2) for k. Since both relations are obvious for $k = 0$, we only need to verify (6.2) and (6.3) for all k with $1 \le k < m$. Fix such an index k.

The relation (6.3) follows from our assumption that $q_1 \in \mathcal{V}$:

$$\overline{w_{k+1} \dots w_m} = \overline{\beta_{k+1} \dots \beta_m} < \overline{\alpha_{k+1} \dots \alpha_m} \le \alpha_1 \dots \alpha_{m-k} = w_1 \dots w_{m-k}.$$

Since $1 \le m - k < m$, we also have

$$\overline{w_{m-k+1} \dots w_m} < w_1 \dots w_k.$$

Using Lemma 6.4 we obtain

$$\begin{aligned} w_{k+1} \dots w_{k+m} &= w_{k+1} \dots w_m \overline{w_1 \dots w_k} \\ &\le w_1 \dots w_{m-k} \overline{w_1 \dots w_k} \\ &< w_1 \dots w_{m-k} w_{m-k+1} \dots w_m, \end{aligned}$$

from which (6.2) follows.

(ii) The sequence (γ_i) is an expansion of 1 in base q_2. It remains to show that

$$\gamma_{k+1} \gamma_{k+2} \dots < w_1 w_2 \dots \quad \text{whenever} \quad \gamma_k < w_1. \tag{6.4}$$

If $1 \le k < m$, then (6.4) follows from

$$\gamma_{k+1} \dots \gamma_{k+m} = w_{k+1} \dots w_{k+m} < w_1 \dots w_m.$$

If $k = m$, then (6.4) follows from $\gamma_{m+1} = \overline{w_1} = 0 < w_1$ (note that $m > 1$).
If $m < k < 2m$, then

$$\gamma_{k+1} \dots \gamma_{2m} = \overline{\beta_{k-m+1} \dots \beta_m^-} \le w_1 \dots w_{2m-k}.$$

Hence

$$\gamma_{k+1} \gamma_{k+2} \dots = \gamma_{k+1} \dots \gamma_{2m} 0^\infty < w_1 w_2 \dots,$$

because (w_i) is infinite. Finally, if $k \ge 2m$, then $\gamma_{k+1} = 0 < w_1$. $\square$

Proof of Proposition 6.1 (ii). It follows from Lemma 6.5 that $\mathcal{V} \setminus \mathcal{U} \subset L$. If $q_2 \sim (n0)^\infty$ for some $n \in \mathbb{N}$, then (n, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Hence $\mathbb{N} \subset L$. It remains to show that $(L \setminus \mathbb{N}) \cap \mathcal{U} = \varnothing$.

If (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$ with $q_2 \sim (\alpha_i)$ and $q_1 \in L \setminus \mathbb{N}$, then by Proposition 6.1 (i) and Lemma 6.3, $q_1 \sim (\alpha_1 \dots \alpha_k^-)^\infty$ for some $k \ge 2$. Since $\alpha_1 \dots \alpha_k 0^\infty$ is another expansion of 1 in base q_1, we have $q_1 \notin \mathcal{U}$. $\square$

Recall from Section 1 that for $q > 1$, $\mathcal{U}_q'$ and $\mathcal{V}_q'$ denote the sets of quasi-greedy expansions in base q of the numbers $x \in \mathcal{U}_q$ and $x \in \mathcal{V}_q$ respectively.

Lemma 6.6. *Let (q_1, q_2) be a connected component of $(1,\infty)\setminus\mathcal{V}$ and suppose that $q_1 \in \mathcal{V}\setminus\mathcal{U}$. Then*

$$\mathcal{U}'_{q_2} = \mathcal{V}'_{q_1}.$$

Proof. First of all, note that $\lceil q_1 \rceil = \lceil q_2 \rceil$ because $q_1 \notin \mathbb{N}$ by assumption. Hence the conjugate bars with respect to q_1 and q_2 have the same meaning.

Let us write again

$$q_2 \sim (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty$$

where k is chosen to be minimal. Suppose that a sequence $(c_i) \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ is univoque in base q_2, i.e.,

$$c_{n+1}c_{n+2}\ldots < (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty \quad \text{whenever} \quad c_n < \alpha_1 \tag{6.5}$$

and

$$\overline{c_{n+1}c_{n+2}\ldots} < (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty \quad \text{whenever} \quad c_n > 0. \tag{6.6}$$

If $c_n < \alpha_1$, then by (6.5),

$$c_{n+1}\ldots c_{n+k} \le \alpha_1 \ldots \alpha_k.$$

If we had

$$c_{n+1}\ldots c_{n+k} = \alpha_1 \ldots \alpha_k,$$

then

$$c_{n+k+1}c_{n+k+2}\ldots < (\overline{\alpha_1 \ldots \alpha_k}\alpha_1 \ldots \alpha_k)^\infty,$$

and by (6.6) (note that in this case $c_{n+k} = \alpha_k > 0$),

$$c_{n+k+1}c_{n+k+2}\ldots > (\overline{\alpha_1 \ldots \alpha_k}\alpha_1 \ldots \alpha_k)^\infty,$$

a contradiction. Hence

$$c_{n+1}\ldots c_{n+k} \le \alpha_1 \ldots \alpha_k^-.$$

Note that $c_{n+k} < \alpha_1$ in case of equality. It follows by induction that

$$c_{n+1}c_{n+2}\ldots \le (\alpha_1 \ldots \alpha_k^-)^\infty.$$

Since a sequence (c_i) satisfying (6.5) and (6.6) is infinite unless $(c_i) = 0^\infty$, we conclude from Proposition 2.4 and Lemma 6.3 that (c_i) is the quasi-greedy expansion of some x in base q_1. Repeating the above argument for the sequence $\overline{c_1c_2\ldots}$, which is also univoque in base q_2, we conclude that $(c_i) \in \mathcal{V}'_{q_1}$. The reverse inclusion follows from the fact that the map $q \mapsto (\alpha_i(q))$ is strictly increasing. $\square$

Lemma 6.7. *Let (q_1, q_2) be a connected component of $(1,\infty)\setminus\mathcal{V}$ and suppose that $q_1 \in \mathcal{V}\setminus\mathcal{U}$. If $q \in (q_1, q_2]$, then*

(i) $\mathcal{U}'_q = \mathcal{V}'_{q_1}$;

(ii) *$\mathcal{U}_q$ contains isolated points if and only if $q_1 \in \mathcal{V}\setminus\overline{\mathcal{U}}$. Moreover, if $q_1 \in \mathcal{V}\setminus\overline{\mathcal{U}}$, then each sequence $(a_i) \in \mathcal{V}'_{q_1}\setminus\mathcal{U}'_{q_1}$ is the expansion in base q of an isolated point of $\mathcal{U}_q$ and each sequence $(c_i) \in \mathcal{U}'_{q_1}$ is the expansion in base q of an accumulation point of $\mathcal{U}_q$.*

Proof. (i) Note that

$$\mathcal{U}'_q \subset \mathcal{U}'_r \quad \text{and} \quad \mathcal{V}'_q \subset \mathcal{U}'_r \quad \text{if } 1 < q < r \text{ and } \lceil q \rceil = \lceil r \rceil. \tag{6.7}$$

It follows from Lemma 6.6 and (6.7) that $\mathcal{U}'_q = \mathcal{V}'_{q_1}$ for all $q \in (q_1, q_2]$.

(ii) We need the following observation (valid for all $q > 1$) which is a consequence of Lemmas 3.1 and 3.2:

If $x \in J_q$ has an infinite greedy expansion, then a sequence (x_i) with elements in J_q converges to x if and only if the greedy expansion of x_i converges (coordinate-wise) to the greedy expansion of x as $i \to \infty$. Moreover, $x_i \downarrow 0$ if and only if the greedy expansion of x_i converges (coordinate-wise) to the sequence 0^∞ as $i \to \infty$.

First assume that $q_1 \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Let $x \in \mathcal{V}_{q_1} \setminus \mathcal{U}_{q_1}$ and denote the quasi-greedy expansion of x in base q_1 by (a_i). Since each element in $\mathcal{V}_{q_1} \setminus \mathcal{U}_{q_1}$ is an isolated point of $\mathcal{V}_{q_1}$ (see Theorem 1.4 (ii)), there exists a positive integer n such that the quasi-greedy expansion in base q_1 of any element in $\mathcal{V}_{q_1} \setminus \{x\}$ does *not* start with $a_1 \dots a_n$. Since $(a_i) \in \mathcal{V}'_{q_1} = \mathcal{U}'_q$, it follows from the above observation that the sequence (a_i) is the unique expansion in base q of an isolated point of $\mathcal{U}_q$. If $x \in \mathcal{U}_{q_1}$, then there exists a sequence of numbers (x_i) with $x_i \in \mathcal{V}_{q_1} \setminus \mathcal{U}_{q_1}$ such that the quasi-greedy expansions of the numbers x_i converge to the unique expansion of x, as can be seen from the proof of Theorem 1.4 (iib) (which in turn relies on the proof of Theorem 1.3 (iib)). Hence the unique expansion of x in base q_1 is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$.

Next assume that $q_1 \in \overline{\mathcal{U}} \setminus \mathcal{U}$. It follows from Theorem 1.3 that the set $\mathcal{U}_{q_1}$ has no isolated points. Hence for each $x \in \mathcal{U}_{q_1}$ there exists a sequence of numbers (x_i) with $x_i \in \mathcal{U}_{q_1} \setminus \{x\}$ such that $x_i \to x$. In view of the above observation, the unique expansions of the numbers x_i converge to the unique expansion of x. Therefore the unique expansion of x in base q_1 is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$. If $x \in \mathcal{V}_{q_1} \setminus \mathcal{U}_{q_1} = \overline{\mathcal{U}_{q_1}} \setminus \mathcal{U}_{q_1}$, then there exists a sequence (x_i) of numbers in $\mathcal{U}_{q_1}$ such that the unique expansions of the numbers x_i converge to the quasi-greedy expansion (a_i) of x, as follows from the proof of Theorem 1.3 (i). Hence, also in this case, (a_i) is the unique expansion in base q of an accumulation point of $\mathcal{U}_q$. Since $\mathcal{U}'_q = \mathcal{V}'_{q_1}$, this completes the proof. $\square$

Lemma 6.8. *Let (q_1, q_2) be a connected component of $(1, \infty) \setminus \mathcal{V}$ and suppose that $q_1 \in \mathbb{N}$. If $q \in (q_1, q_2]$, then $\mathcal{U}'_q = \mathcal{U}'_{q_2}$ and $\mathcal{U}_q$ contains isolated points if and only if $q_1 \in \{1, 2\}$.*

Proof. Note that if $q_1 = n \in \mathbb{N}$, then $q_2 \sim (n0)^\infty$. Suppose that $q \in (n, q_2]$. We leave the verification of the following statements to the reader.
A sequence $(a_i) \in \{0, \dots, n\}^{\mathbb{N}}$ belongs to $\mathcal{U}'_q$ if and only if for all $j \in \mathbb{N}$,

$$a_j < n \Longrightarrow a_{j+1} < n$$

and

$$a_j > 0 \Longrightarrow a_{j+1} > 0.$$

In particular we see that

$$\mathcal{U}'_q = \mathcal{U}'_{q_2}.$$

If $n = 1$, then $\mathcal{U}'_q = \{0^\infty, 1^\infty\}$. If $n = 2$, then

$$\mathcal{U}'_q = \{0^\infty, 2^\infty\} \cup \bigcup_{n=0}^{\infty} \{0^n 1^\infty, 2^n 1^\infty\}.$$

Hence, if $n = 2$, then $\mathcal{U}_q$ is countable and all elements of $\mathcal{U}_q$ are isolated, except for its endpoints. If $n \geq 3$, then $\mathcal{U}_q$ has no isolated points. $\square$

Proposition 6.9. *Let $q > 1$ be a real number.*

(i) *If $q \in \overline{\mathcal{U}}$, then q is neither stable from below nor stable from above.*
(ii) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then q is stable from below, but not stable from above.*
(iii) *If $q \notin \mathcal{V}$, then q is stable.*

Proof. (i) As mentioned at the beginning of this section, if $q \in \overline{\mathcal{U}}$, then there exists a sequence $(q_m)_{m \geq 1}$ with numbers $q_m \in \mathcal{V} \setminus \overline{\mathcal{U}}$, such that $q_m \uparrow q$. Since

$$\mathcal{U}'_{q_m} \subsetneq \mathcal{V}'_{q_m} \subset \mathcal{U}'_q,$$

q is not stable from below. If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then q is not stable from above because

$$\mathcal{U}'_q \subsetneq \mathcal{V}'_q \subset \mathcal{U}'_s \tag{6.8}$$

for any $s \in (q, \lceil q \rceil]$. If $q \in \{2, 3, \ldots\}$, then q is not stable from above because the sequence q^∞ belongs to $\mathcal{U}_s' \setminus \mathcal{U}_q'$ for any $s > q$.

(ii) and (iii) If $q \notin \overline{\mathcal{U}}$, then $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. From Proposition 6.1 and Lemmas 6.7 and 6.8 we conclude that q is stable from below. Note that $q = q_2$ if and only if $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Hence, if $q \notin \mathcal{V}$, then q is also stable from above. If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then q is not stable from above because (6.8) holds for any $s \in (q, \lceil q \rceil]$. □

Remark. The issue of (non-)stability of bases is further explored in [DV2].

Proof of Theorem 1.6. (i) This is the content of Proposition 6.1.

(ii) If $q \in \{2, 3, \ldots\}$, then $\mathcal{U}_q \subsetneq \overline{\mathcal{U}_q} = [0, 1]$. Hence, neither $\mathcal{U}_q$ nor $\overline{\mathcal{U}_q}$ is a Cantor set.

(iii) and (iv) If $q \notin \mathbb{N}$, then $\mathcal{U}_q$ is nowhere dense by a remark following the statement of Theorem 1.5 in Section 1. Hence, if $q \notin \mathbb{N}$, then $\mathcal{U}_q$ is a Cantor set if and only if $\mathcal{U}_q$ is closed and does not contain isolated points.

If $q \in \overline{\mathcal{U}} \setminus \mathbb{N}$, then by Theorem 1.3 the set $\mathcal{U}_q$ is not closed, and $\overline{\mathcal{U}_q}$ has no isolated points, from which part (iii) follows.

Finally, let $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Since $q \notin \overline{\mathcal{U}}$, the set $\mathcal{U}_q$ is closed. It follows from Lemmas 6.7 and 6.8 that $\mathcal{U}_q$ is a Cantor set if and only if $q_1 \in \{3, 4, \ldots\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$. A quick examination of the proof of these lemmas yields the last statement of part (iv). □

Proof of Theorem 1.7. The statements of this theorem readily follow from Proposition 6.1, Lemmas 6.7 and 6.8, and Proposition 6.9. □

Proof of Theorem 1.8. Fix $q \in (1, \infty) \setminus \overline{\mathcal{U}}$. Then $q \in (q_1, q_2]$, where (q_1, q_2) is a connected component of $(1, \infty) \setminus \mathcal{V}$. Let us write

$$q_2 \sim (\alpha_i) = (\alpha_1 \ldots \alpha_k \overline{\alpha_1 \ldots \alpha_k})^\infty,$$

where k is minimal. Let

$$\mathcal{F} = \left\{ j a_1 \ldots a_k \in \{0, \ldots, \alpha_1\}^{k+1} : j < \alpha_1 \quad \text{and} \quad a_1 \ldots a_k \geq \alpha_1 \ldots \alpha_k \right\}.$$

It follows from Lemma 6.7 (i) and the proof of Lemmas 6.6 and 6.8 that a sequence $(c_i) \in \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ belongs to $\mathcal{U}_q'$ if and only if $c_j \ldots c_{j+k} \notin \mathcal{F}$ and $\overline{c_j \ldots c_{j+k}} \notin \mathcal{F}$ for all $j \geq 1$. Therefore, $\mathcal{U}_q'$ is a subshift of finite type. Note that any subshift $S \subset \{0, \ldots, \alpha_1\}^{\mathbb{N}}$ is closed in the topology of coordinate-wise convergence. It remains to show that $\mathcal{U}_q'$ is not closed if $q \in \overline{\mathcal{U}}$. It follows from the proof of Theorem 1.3 (i) that for each $q \in \overline{\mathcal{U}}$ and $x \in \mathcal{V}_q \setminus \mathcal{U}_q$, there exists a sequence (x_i) of numbers in $\mathcal{U}_q$ such that the unique expansions of the numbers x_i converge to the quasi-greedy expansion of x. Hence the set $\mathcal{U}_q'$ is not closed if $q \in \overline{\mathcal{U}}$. □

Proof of Theorem 1.9. (i) Note that $G \sim (10)^\infty$. It follows from the proof of Lemma 6.8 that $\mathcal{U}_q' = \{0^\infty, 1^\infty\}$ for all $q \in (1, G]$.

(ii) Due to the properties of the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ which were mentioned at the beginning of this section, we may write

$$\mathcal{V} \cap (1, q') = \{q_n : n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{V} \cap (2, q'') = \{r_n : n \in \mathbb{N}\},$$

where the q_n's and the r_n's are written in strictly increasing order. Note that $q_1 \sim (10)^\infty$ and $r_1 \sim (20)^\infty$. Moreover, $q_n \uparrow q'$ and $r_n \uparrow q''$. Thanks to (6.7) we only need to verify that the sets $\mathcal{U}_{q_n}$ and $\mathcal{U}_{r_n}$ are countable for each $n \in \mathbb{N}$. It follows from the proof of Lemma 6.8 that $\mathcal{U}_{q_1}$ and $\mathcal{U}_{r_1}$ are countable. Now suppose that $\mathcal{U}_{q_n}$ is countable for some $n \geq 1$. By Lemma 6.6 we have

$$\mathcal{U}_{q_{n+1}}' = \mathcal{V}_{q_n}' = \mathcal{U}_{q_n}' \cup (\mathcal{V}_{q_n}' \setminus \mathcal{U}_{q_n}').$$

According to Theorem 1.4 (ii) the set $\mathcal{V}'_{q_n} \setminus \mathcal{U}'_{q_n}$ is countable, whence $\mathcal{U}_{q_{n+1}}$ is countable as well. It follows by induction that $\mathcal{U}_{q_n}$ is countable for each $n \in \mathbb{N}$. Similarly, $\mathcal{U}_{r_n}$ is countable for each $n \in \mathbb{N}$.

(iii) It follows from Theorem 1.3 that $|\mathcal{U}_{q'}| = 2^{\aleph_0}$ and $|\mathcal{U}_{q''}| = 2^{\aleph_0}$. The relation (6.7) yields that $|\mathcal{U}_q| = 2^{\aleph_0}$ for all $q \in [q', 2] \cup [q'', 3]$. If $q > 3$, then $|\mathcal{U}_q| = 2^{\aleph_0}$ because $\mathcal{U}'_q$ contains all sequences consisting of merely ones and twos. $\square$

We conclude this paper with an example and some remarks.

Example. For any given $k \in \mathbb{N}$ define the numbers $p(k)$ and $q(k)$ by setting

$$p(k) \sim (1^{k-1}0)^\infty \quad \text{and} \quad q(k) \sim (1^k 0^k)^\infty.$$

It follows from Lemma 6.5 and Theorem 1.7 that the sets $(p(k), q(k)]$ are maximal stability intervals. Moreover, it follows from the proof of Theorem 1.8 that a sequence $(c_i) \in \{0,1\}^{\mathbb{N}}$ belongs to $\mathcal{U}'_q$ for $q \in (p(k), q(k)]$ if and only if a zero is never followed by k consecutive ones and a one is never followed by k consecutive zeros. This result was first established by Daróczy and Kátai in [DK1], using a different approach.

From Lemma 6.5 we know that the least element of $\mathcal{V}$ larger than $q(k)$ is given by $r(k)$, where

$$r(k) \sim (1^k 0^{k-1} 1 0^k 1^{k-1} 0)^\infty.$$

Therefore the sets $(q(k), r(k)]$ are also maximal stability intervals. If $q \in (q(k), r(k)]$, then $\mathcal{U}_q$ is not a Cantor set because $q(k) \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Moreover, a number $x \in \mathcal{U}_q$ is an isolated point of $\mathcal{U}_q$ if and only if its unique expansion belongs to $\mathcal{V}'_{q(k)} \setminus \mathcal{U}'_{q(k)}$.

Finally, let $k \geq 3$, and let $s(k)$ be the least element of $\mathcal{V}$ larger than $r(k)$. Fix $q \in (r(k), s(k)]$ and let (c_i) be the unique expansion of a number $x \in \mathcal{U}_q$. Combining Lemma 6.7 with the final remark below, one can "decompose" $\mathcal{U}_q$ as follows:

- x is an isolated point of $\mathcal{U}_q$ if and only if $(c_i) \in \mathcal{V}'_{r(k)} \setminus \mathcal{U}'_{r(k)}$.
- x is an accumulation point of $\mathcal{U}_q$ if and only if $(c_i) \in \mathcal{U}'_{r(k)}$.
- x is a condensation point of $\mathcal{U}_q$ if and only if $(c_i) \in \mathcal{U}'_{q(k)}$.

Remarks.

- Let us now consider the set L' of left endpoints and the set R' of right endpoints of the connected components of $(1, \infty) \setminus \overline{\mathcal{U}}$. We will show that

$$L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad R' \subsetneq \mathcal{U} \setminus \mathbb{N}.$$

Fix a number $q \in (1, \infty) \setminus \overline{\mathcal{U}}$. Let q_1 be the least element of $\mathcal{V}$ satisfying $q_1 \geq q$. Since $q_1 \in \mathcal{V} \setminus \overline{\mathcal{U}}$, it is both a left endpoint and a right endpoint of a connected component of $(1, \infty) \setminus \mathcal{V}$. Hence there exists a sequence $q_1 < q_2 < \cdots$ of numbers in $\mathcal{V} \setminus \overline{\mathcal{U}}$ satisfying

$$(q_i, q_{i+1}) \cap \mathcal{V} = \varnothing \quad \text{for all} \quad i \geq 1.$$

Let β_m be the last nonzero element of the greedy expansion (β_i) of the number 1 in base q_1. We define a sequence (c_i) by induction as follows. First, set

$$c_1 \ldots c_m = \beta_1 \ldots \beta_m.$$

Then, if $c_1 \ldots c_{2^N m}$ is already defined for some nonnegative integer N, set

$$c_{2^N m+1} \ldots c_{2^{N+1} m-1} = \overline{c_1 \ldots c_{2^N m-1}} \quad \text{and} \quad c_{2^{N+1} m} = \overline{c_{2^N m}} + 1.$$

It follows from Lemma 6.5 that the greedy expansion of 1 in base q_n is given by $c_1 \ldots c_{2^{n-1} m} 0^\infty$ $(n \in \mathbb{N})$. Hence (c_i) is an expansion of 1 in base q^* where

$$q^* = \lim_{n \to \infty} q_n.$$

Note that $q^* \in \mathcal{V}$ because $\mathcal{V}$ is closed. The number q^* cannot belong to $\mathcal{V} \setminus \overline{\mathcal{U}}$ because this set is discrete, and it cannot belong to $\overline{\mathcal{U}} \setminus \mathcal{U}$ because $\mathcal{U}$ is closed from above ([KL3]). Hence $q^* \in \mathcal{U}$ and $R' \subset \mathcal{U} \setminus \mathbb{N}$. The set R' is a proper subset of $\mathcal{U} \setminus \mathbb{N}$ because the latter set is uncountable.

Now let ℓ_1 be the largest element of $\mathcal{V} \cup \{1\}$ that is smaller than q_1. Let us also write $\ell_1 \sim (\eta_i)$ and $q_1 \sim (\alpha_i)$. It follows from Lemma 6.3 and the remark preceding Lemma 6.2 that (η_i) has a smaller period than the least period of (α_i). Hence there exists a finite set of numbers $\ell_k < \cdots < \ell_1$ in $\mathcal{V} \cup \{1\}$, such that for i with $1 \leq i < k$,

$$(\ell_{i+1}, \ell_i) \cap \mathcal{V} = \varnothing,$$

and such that ℓ_k is a left endpoint of a connected component of $(1, \infty) \setminus \mathcal{V}$, but not a right endpoint. This means that

$$\ell_k \in \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U}) \quad \text{and} \quad (\ell_k, q) \cap \overline{\mathcal{U}} = \varnothing.$$

Hence $\ell_k \in \overline{\mathcal{U}} \cup \{1\}$ and therefore $\ell_k \in L'$. We may thus conclude that $L' \subset \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$. On the other hand, $L \cap \overline{\mathcal{U}} \subset L'$ because $\overline{\mathcal{U}} \subset \mathcal{V}$. Taking into account that $1 \in L'$, we deduce from Proposition 6.1 that $L' = \mathbb{N} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$.

- The analysis of the preceding remark enables us also to determine for each $n \in \mathbb{N}$ the least element $q^{(n)}$ of the set $\mathcal{U} \cap (n, n+1)$:

  Fix $n \in \mathbb{N}$ and let q be the least element of $\mathcal{V} \cap (n, n+1)$. Then $q \sim (n0)^\infty$ and the greedy expansion (β_i) of 1 in base q is given by $n10^\infty$. The sequence (c_i) constructed in the preceding remark with $m = 2$ and $c_1 c_2 = n1$ is the unique expansion of the number 1 in base $q^{(n)}$.
- In [KL2] it was shown that for each $n \in \mathbb{N}$, there exists a smallest number $r^{(n)} > 1$ for which exactly one sequence $(c_i) \in \{0, 1, \ldots, n\}^{\mathbb{N}}$ satisfies the equality

$$\sum_{i=1}^{\infty} c_i (r^{(n)})^{-i} = 1.$$

  Although this might appear as an equivalent definition of the numbers $q^{(n)}$, there is a subtle difference: it is not required that $r^{(n)}$ belongs to $(n, n+1)$. It can be seen from the results in [KL2] that $r^{(n)} = q^{(n)}$ if and only if $n \in \{1, 2\}$. If $n > 2$, then $r^{(n)} < n < q^{(n)}$.
- Finally, we determine the condensation points of $\mathcal{U}_q$ for $q > 1$. If $q \in \overline{\mathcal{U}}$, then each element of $\mathcal{U}_q$ is a condensation point of $\mathcal{U}_q$ because the set $\overline{\mathcal{U}_q}$ is perfect and $\overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ is countable. Henceforth assume that $q \in (1, \infty) \setminus \overline{\mathcal{U}}$. Let (p_1, p_2) be the connected component of $(1, \infty) \setminus \overline{\mathcal{U}}$ containing q, and let r be the least element of $(p_1, p_2) \cap \mathcal{V}$. Since $\mathcal{U}_q$ is a closed set, it can be written uniquely as a disjoint union of a countable set C and a perfect set P consisting precisely of the condensation points of $\mathcal{U}_q$. We claim that

$$P = \left\{ \sum_{i=1}^{\infty} c_i q^{-i} : (c_i) \in \mathcal{U}_r' \right\},$$

  except when $p_1 \in \{1, 2\}$, in which case $P = \varnothing$. Indeed, if $p_1 \notin \{1, 2\}$, then $\mathcal{U}_r$ is a Cantor set by Theorem 1.6 (iv). It follows in particular that the set $\mathcal{U}_r$ is perfect and hence consists entirely of condensation points. Note that $\mathcal{U}_r' \subset \mathcal{U}_q'$ because $(p_1, r]$ is a stability interval and $\lceil r \rceil = \lceil q \rceil$. If $q \in (p_1, r]$, then $\mathcal{U}_r' = \mathcal{U}_q'$. If $q \in (r, p_2)$, then $[r, q) \cap \mathcal{V}$ is a finite subset of $\mathcal{V} \setminus \overline{\mathcal{U}}$, as follows from the first remark above. Moreover, if we write $[r, q) \cap \mathcal{V} = \{r_1, \ldots, r_m\}$ where $r_1 < \cdots < r_m$, then by applying Lemma 6.7 (i) (m times), we get

that

$$\mathcal{U}'_q = \mathcal{U}'_r \cup \bigcup_{\ell=1}^{m} (\mathcal{V}'_{r_\ell} \setminus \mathcal{U}'_{r_\ell}).$$

Hence $\mathcal{U}'_q \setminus \mathcal{U}'_r$ is countable by Theorem 1.4 (ii).

Now let $x \in \mathcal{U}_q$ and let (c_i) be its unique expansion in base q. Suppose first that (c_i) belongs to $\mathcal{U}'_r$. Let W be an arbitrary neighborhood of x. There exists an index N such that each univoque sequence in base q starting with the block $c_1 \dots c_N$ is the unique expansion in base q of a number in W. Applying Lemmas 3.1 and 3.2 to greedy expansions in base r and using the fact that $\mathcal{U}_r$ is perfect and $\mathcal{U}'_r \subset \mathcal{U}'_q$, we conclude that x is a condensation point of $\mathcal{U}_q$. If (c_i) does not belong to $\mathcal{U}'_r$, then there exists an index N such that no sequence in $\mathcal{U}'_r$ starts with $c_1 \dots c_N$ because $\mathcal{U}_r$ is closed. Applying Lemmas 3.1 and 3.2 to greedy expansions in base q and using the fact that $\mathcal{U}'_q \setminus \mathcal{U}'_r$ is countable, we conclude that x is not a condensation point of $\mathcal{U}_q$. If $p_1 \in \{1, 2\}$, then $\mathcal{U}_q$ is countable by Theorem 1.9 and thus $P = \varnothing$.

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